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Midterm 2 Overview and Review

This Midterm 2 Overview and Review serves three purposes:

- To provide an overview of the format of Midterm 2.
- To provide a detailed list of topics you are expected to know for Midterm 2 and can expect to be tested on.
- To provide some practice problems to aid your studying.

Exam Format

(I) Unless otherwise arranged, you have two hours to complete the exam.

(II) Calculators are **not permitted** for use during the exam.

(III) The exam is **closed book**: notes, textbooks, computers, mobile devices, listening devices are **not permitted** for use during the exam.

(IV) There will be a total of five problems on the exam. To receive full credit you will need to provide correct and complete solutions to all five problems.

(V) There will be one problem consisting of several True/False subproblems.

(VI) There will be one short-answer problem consisting of several short, computational subproblems.

(VII) There will be three long-answer problems. Each long-answer problem may have multiple parts.

(VIII) A student with a solid grasp of the material and excellent problem-solving ability should be able to complete the exam in < 1 hour.

Outline of Topics to Know

Theory

You should know all definitions/concepts from lecture (and used in homework) precisely. In particular, you should know the definitions of the following terms/concepts and how they are used:

(a) The chain rule.

(b) The derivative of $\ln(x)$.

(c) The notion of logarithmic differentiation.

(d) The definition of local maximum/minimum.

(e) The definition of global/maximum of a function on an interval.

(f) The definition of critical/inflection points.

(g) The local linearisation and linear approximation.

(h) What it means for a function to be differentiable at a point.

(i) What it means for a function to be differentiable on an interval.

(j) How to model optimisation problems.

(k) How to model related rates problems.

(l) What it means for $g(x)$ to dominate $f(x)$ and how this relates to rates of growth.

In addition, **you should know examples of functions that are differentiable, and why this is the case.**.

Major Results

You should thoroughly understand all of the major results we've discussed this semester; this includes all of the theorems and propositions we've discussed in class. This means, in particular, having a solid understanding of each theorem's hypothesis and, if I have discussed the necessity of a certain hypotheses, you should understand why. Here are some of the big results we've discussed so far.

(i) You should know the relationship between a function and its local linearisation at a point.

(ii) You should know Fermat's Theorem and its consequences for classifying local maxima/minima.

(iii) You should know the Extreme Value Theorem and its consequences for global extrema problems.

(iv) You should know the Mean Value Theorem.

(v) You should know the Increasing/Decreasing/Constant Function Theorems.

(vi) You should know the Racetrack Principle.

(vii) You should know the First/Second Derivative Tests and their implications.

(viii) You should know L'Hopital's Rule and when it can be applied.

Computation

In general, you should know how to work problems connected to each key concept discussed in the previous section. As I said before, if you understand all of the lecture material and all of the homework (both the "turn in" and "do not turn in" problems), are able to work quickly and efficiently, you should perform well on the exam. Here is an incomplete list of things we have done so far this semester:

1. You should know how to compute derivatives of a broad collection of functions using the power/exponential/p rules.
2. You should know how to use the local linearisation to estimate a function.
3. You should know how to apply the Racetrack Principle.
4. You should know how to classify local/global maxima/minima using First/Second Derivative Tests.
5. You should know how to the determine the nature of critical points.
6. You should know how to determine inflection points.
7. You should know how to set up optimisation problems.
8. You should know how to set up related rates problems.
9. You should know how to determine limits using L'Hopital's Rule.

Review Problems

The following problems will provide you with good practice for the midterm. You should also attempt the *Do not turn in* problems from homework. If you want more problems then please feel free to ask.

Section 3.10 1-9, 30-37
Chapter 3 Summary (*p.180-184*) 1-81, 97, 105
Chapter 4 Summary (*p.260-266*) 3-6, 10-13, 17-22, 24, 27-32, 35, 43-51, 56, 65-69, 71, 81, 94

Logarithmic Differentiation Using logarithmic differentiation compute the derivative of the function.

1. $y = \sqrt{\dfrac{x-1}{x^4+1}}$
2. $y = \sqrt{x}^x$
3. $y = x^{\cos(x)}$
4. $y = (\sin(x))^{\ln(x)}$
5. $y = (\tan(x))^{1/x}$

Solution:

1. Let $y = \sqrt{\frac{x-1}{x^4+1}} = \left(\frac{x-1}{x^4+1}\right)^{1/2}$. Then,

$$\ln(y) = \ln\left(\frac{x-1}{x^4+1}\right)^{1/2} = \frac{1}{2}\left(\ln(x-1) - \ln(x^4+1)\right)$$

Take the derivative on both sides with respect to x:

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\frac{1}{x-1} - \frac{2x^3}{x^4+1}$$

$$\Longrightarrow \qquad \frac{dy}{dx} = y\left(\frac{1}{2}\frac{1}{x-1} - \frac{2x^3}{x^4+1}\right) = \sqrt{\frac{x-1}{x^4+1}}\left(\frac{1}{2}\frac{1}{x-1} - \frac{2x^3}{x^4+1}\right)$$

2. Let $y = \sqrt{x}^x = (x^{1/2})^x = x^{x/2}$. Then,

$$\ln(y) = \ln(x^{x/2}) = \frac{x}{2}\ln(x)$$

Take the derivative on both sides with respect to x:

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\ln(x) + \frac{x}{2}\frac{1}{x} = \frac{1}{2}(\ln(x) + 1)$$

$$\Longrightarrow \qquad \frac{dy}{dx} = \frac{y}{2}(\ln(x) + 1) = \frac{\sqrt{x}^x}{2}(\ln(x) + 1)$$

3. Let $y = x^{\cos(x)}$. Then,

$$\ln(y) = \ln(x^{\cos(x)}) = \cos(x)\ln(x)$$

Take the derivative on both sides with respect to x:

$$\frac{1}{y}\frac{dy}{dx} = -\sin(x)\ln(x) + \frac{\cos(x)}{x}$$

$$\Longrightarrow \qquad \frac{dy}{dx} = y(-\sin(x)\ln(x) + \frac{\cos(x)}{x}) = x^{\cos(x)}(-\sin(x)\ln(x) + \frac{\cos(x)}{x})$$

4. Let $y = (\sin(x))^{\ln(x)}$. Then,

$$\ln(y) = \ln(\sin(x)^{\ln(x)}) = \ln(x)\ln(\sin(x))$$

Take the derivative on both sides with respect to x:

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x}\ln(\sin(x)) + \ln(x)\frac{\cos(x)}{\sin(x)}$$

$$\implies \qquad \frac{dy}{dx} = y\left(\frac{1}{x}\ln(\sin(x)) + \ln(x)\frac{\cos(x)}{\sin(x)}\right) = \sin(x)^{\ln(x)}\left(\frac{1}{x}\ln(\sin(x)) + \ln(x)\frac{\cos(x)}{\sin(x)}\right)$$

5. Let $y = \tan(x)^{1/x}$. Then,

$$\ln(y) = \ln(\tan(x)^{1/x}) = \frac{1}{x}\ln(\tan(x))$$

Take the derivative on both sides with respect to x:

$$\frac{1}{y}\frac{dy}{dx} = -\frac{1}{x^2}\ln(\tan(x)) + \frac{1}{x}\frac{1}{\tan(x)}\sec^2(x)$$

$$\implies \qquad \frac{dy}{dx} = y\left(-\frac{1}{x^2}\ln(\tan(x)) + \frac{1}{x}\frac{1}{\tan(x)}\sec^2(x)\right) = \tan(x)^{1/x}\left(-\frac{1}{x^2}\ln(\tan(x)) + \frac{1}{x}\frac{1}{\tan(x)}\sec^2(x)\right)$$