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Practice Midterm 2

Disclaimer: This Practice Midterm consists of problems of a similar difficulty as will be on the actual midterm. However, **problems on the actual midterm may or may not be quite different in nature, and may or may not focus on different course material**. However, the actual midterm will have a similar format: one true/false problem, one short-answer problem, three long-answer problems.

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1. True/False (no justification required)

   (a) $\lim_{x\to 0^+} \frac{\ln(x^2)}{x} = 1$

   (b) Let $f(x) = e^x + 5$, where $-1 \leq x \leq 10$. Then, $x = 10$ is a local maximum of $f(x)$.

   (c) If $f''(p) = 0$ then p is an inflection point.

   (d) The linear approximation of $f(x) = \cos(x)$ at $x = \pi/2$ is 0.

   (e) Let $y = \sin(\cos(x))$. Then, $\frac{dy}{dx} = \sin(\cos(x)) + \cos(\sin(x))$.

   Solution:

   (a) False: the limit DNE by L'Hopital.

   (b) False: by definition, local maxima can't occur at endpoints.

   (c) False: $f(x) = x^4$, $p = 0$, provides a counterexample.

   (d) False: it's $L(x) = -x + \frac{\pi}{2}$.

   (e) False: incorrect application of Chain Rule.

2. (a) Let $g(t) = e^{\cos(t^3)}$. Determine $g'(t)$.

   (b) Let $y = (1-2x)^{e^x}$. Using logarithmic differentiation, determine $\frac{dy}{dx}$.

   Solution:

   (a) Apply Chain Rule (twice):

   $$g'(t) = e^{cos(t^3)} \cdot \frac{d}{dt}(\cos(t^3)) = e^{\cos(t^3)}(-\sin(t^3)3t^2)$$

   (b) Let $y = (1-2x)^{e^x}$. Then, apply $\ln(\cdot)$ to both sides gives

   $$\ln(y) = \ln\left((1-2x)^{e^x}\right) = e^x \ln(1-2x)$$

   Take the derivative of both sides with respect to x: the Chain Rule gives

   $$\frac{1}{y}\frac{dy}{dx} = e^x \ln(1-2x) - \frac{2e^x}{1-2x}$$

   $$\implies \qquad \frac{dy}{dx} = ye^x\left(\ln(1-2x) - \frac{2}{1-2x}\right) = e^x\left((1-2x)^{e^x}\right)\left(\ln(1-2x) - \frac{2}{1-2x}\right)$$

3. A piece of wire having length 10m is cut into two pieces. One piece is bent into a square and the other is bent into an equilateral triangle.

(a) Let x be the side length of the square. Show that the total area A of the square and triangle can be expressed as

$$A = x^2 + \frac{\sqrt{3}}{4}\left(\frac{10-4x}{3}\right)^2$$

Hint: the area of an equilateral triangle having side length a is $\frac{\sqrt{3}}{4}a^2$.

(b) Determine the side length of the square giving the largest possible area A.

Solution:

(a) We have

$$A = \text{area of square } + \text{ area of equilateral triangle}$$

Denote by x the side length of the square. Then, the piece of wire used to make the square has length $4x$ metres. Therefore, the length of wire used to make the equilateral triangle is $10-4x$ metres. Hence, the side length of the equilateral triangle is $\frac{10-4x}{3}$. Combining this information together and using the formula for the area of an equilateral triangle gives

$$A = x^2 + \frac{\sqrt{3}}{4}\left(\frac{10-4x}{3}\right)^2$$

(b) We compute

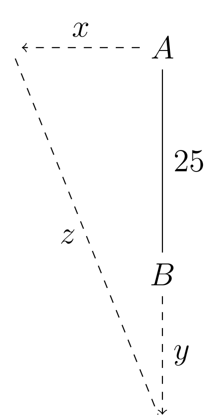
$$\frac{dA}{dx} = 2x + \frac{\sqrt{3}}{4} \times 2\left(\frac{10-4x}{3}\right) \times (-4) = 2x - \frac{2}{\sqrt{3}}(10-4x) = (2+8/\sqrt{3})x - 20/\sqrt{3}$$

Hence, $\frac{dA}{dx} = 0 \implies x = \frac{10}{\sqrt{3}+4}$. Thus, the largest possible area is

$$A = \left(\frac{10}{\sqrt{3}+4}\right)^2 + \frac{\sqrt{3}}{4}\left(\frac{10}{3} - \frac{40}{3(\sqrt{3}+4)}\right)^2 = 3.4828\ldots$$

4. At 1pm ship A is 25km due north of ship B. If Ship A is sailing west at a rate of 16km/h and ship B is sailing south at 20km/h, find the rate at which the distance between the two ships is changing at 1.30pm.

Solution: We have the following diagram representing the situation:



Thus, we know that $\frac{dx}{dt} = 16$ and $\frac{dy}{dt} = 20$. Moreover,

$$z^2 = x^2 + (25+y)^2$$

We want to determine $\frac{dz}{dt}$ =??? at 1.30pm i.e. when $x = 8$ and $y = 10$ (since this is the distance the ships have travelled in 30 minutes). Differentiating both sides of the above equation with respect to t gives

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2(25+y)\frac{dy}{dt}$$

Substituting the known information, and using that $z = \sqrt{35^2+8^2} = 35.9026\ldots$ when $x = 8, y = 10$, gives

$$\frac{dz}{dt} = \frac{8}{\sqrt{35^2+8^2}} \times 16 + \frac{35}{\sqrt{35^2+8^2}} \times 20 = \frac{828}{\sqrt{1289}} = 23.0623\ldots$$

At 1.30pm the distance between the two ships is changing at a rate of approx. 23.06 km/h.

5. Let $f(x) = xe^{-x^2}$.

   (a) Determine the local maxima/minima of $f(x)$.

   (b) Determine the inflection points of $f(x)$.

   (c) Using L'Hopital's Rule, show that $\lim\limits_{x\to\infty} f(x) = 0$ and $\lim_{x\to-\infty} f(x) = 0$.

   (d) Using what you've found, sketch the graph of $f(x)$.

Solution:

(a) We compute

$$f'(x) = e^{-x^2} - 2x^2e^{-x^2} = e^{-x^2}(1-2x^2)$$

and $f'(x) = 0 \implies 1 - 2x^2 = 0 \implies x = \pm\frac{1}{\sqrt{2}}$. Critical points at $x = \pm\frac{1}{\sqrt{2}}$. Apply First Derivative Test:

- $x < -\frac{1}{\sqrt{2}}$: $f'(x) = e^{-x^2}(1-2x^2) = (>0)\cdot(<0) < 0 \quad (-)$
- $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$: $f'(x) = e^{-x^2}(1-2x^2) = (>0)\cdot(>0) > 0 \quad (+)$
- $x > \frac{1}{\sqrt{2}}$: $f'(x) = e^{-x^2}(1-2x^2) = (>0)\cdot(<0) < 0 \quad (-)$

Hence, by the First Derivative Test $x = -\frac{1}{\sqrt{2}}$ is a local minimum and $x = \frac{1}{\sqrt{2}}$ is a local maximum.
Note: you could also use SDT to classify extrema.

(b) We compute the second derivative

$$f''(x) = -2xe^{-x^2}(1-2x^2) - 4xe^{-x^2} = -2xe^{-x^2}(3-2x^2)$$

Possible inflection points when $f''(x) = 0 \implies x = 0, \pm\sqrt{\frac{3}{2}}$. Need to check whether $f''(x)$ changes sign:

- $x < -\sqrt{\frac{3}{2}}$: $f''(x) = -2xe^{-x^2}(3-2x^2) = (>0)\cdot(<0) < 0 \quad (-)$
- $-\sqrt{\frac{3}{2}} < x < 0$: $f''(x) = -2xe^{-x^2}(3-2x^2) = (>0)\cdot(>0) > 0 \quad (+)$
- $0 < x < \sqrt{\frac{3}{2}}$: $f''(x) = -2xe^{-x^2}(3-2x^2) = (<0)\cdot(>0) < 0 \quad (-)$
- $x > \sqrt{\frac{3}{2}}$: $f''(x) = -2xe^{-x^2}(3-2x^2) = (<0)\cdot(<0) > 0 \quad (+)$

Since the second derivative changes signs at $x = 0, \pm\sqrt{\frac{3}{2}}$, they are inflection points.

(c) Using L'Hopital's Rule

$$\lim_{x\to-\infty} \frac{x}{e^{x^2}} = \lim_{x\to-\infty} \frac{1}{2xe^{x^2}} = -0 = 0$$

$$\lim_{x\to\infty} \frac{x}{e^{x^2}} = \lim_{x\to\infty} \frac{1}{2xe^{x^2}} = 0$$

(d) Using the information determined above: