Contact: gwmelvin@colby.edu

Practice Midterm 1: Solution

Disclaimer: This Practice Midterm consists of problems of a similar difficulty as will be on the actual midterm. However, **problems on the actual midterm may or may not be quite different in nature, and may or may not focus on different course material**. However, the actual midterm will have a similar format: one true/false problem, one short-answer problem, three long-answer problems.

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1. True/False (no justification required)

   (a) If $\lim_{x\to a} f(x)$ exists then $f'(a)$ exists.

   (b) If $\arcsin(x) = 5$ then $x = \sin(5)$.

   (c) If $f(5) = -1$ then $\lim_{x\to 5} f(x) = -1$.

   (d) $\frac{d}{dx}x^e = x^e$.

   (e) Let $f(x), g(x)$ be functions that are continuous at $x = 1$. If $g(1) = 1$ and $f(1) = -1$ then $\lim_{x\to 1} f(g(1)) = -1$.

   Solution:

   (a) False

   (b) True

   (c) False

   (d) False

   (e) True

2. Compute $f'(x)$ for the given function $f(x)$.

   (a) $f(x) = (5x^2 + 2\sqrt[3]{x^2})e^x + \pi^e \sin(x)$

   (b) $f(x) = \frac{x^{12/5} - 3x + 10}{e^x + e^{2x}}$

   Solution:

   (a) Note: π^e is a constant.

   $$f(x) = uv + \pi^e \sin(x), \quad \text{where } u = 5x^2 + 2\sqrt[3]{x^2} = 5x^2 + 2x^{2/3},\ v = e^x$$

   $$\implies \quad f'(x) = u'v + uv' + \pi^e \cos(x), \quad \text{by Product Rule and } \frac{d}{dx}\sin(x) = \cos(x)$$

   Compute, using Power Rule,

   $$u' = 10x + \frac{4}{3}x^{-1/3}, \quad v' = e^x$$

   Hence,

   $$f'(x) = e^x(5x^2 + 10x + 2x^{2/3} + \frac{4}{3}x^{-1/3}) + \pi^e \cos(x)$$

(b) We use Quotient Rule

$$f(x) = \frac{u}{v}, \quad \text{where } u = x^{12/5} - 3x + 10,\ v = e^x + e^{2x}$$

Then,

$$f'(x) = \frac{u'v - uv'}{v^2}$$

Compute

$$u' = \frac{12}{5}x^{7/5} - 3, \quad v' = e^x + 2e^{2x}$$

Therefore,

$$f'(x) = \frac{((12/5)x^{7/5} - 3)(e^x + e^{2x}) - (x^{12/5} - 3x + 10)(e^x + 2e^{2x})}{(e^x + e^{2x})^2}$$

3. Let k, l be constants.

(a) Using limit laws, or otherwise, determine

$$\lim_{x \to 1} \frac{x^2 + kx - 2}{x^2 + 5x - 5}$$

Note: your answer will be in terms of k.

(b) Define

$$f(x) = \begin{cases} \frac{x^2+kx-2}{x^2+5x-5}, & x \neq 1 \\ l, & x = 1 \end{cases}$$

Determine all values of k, l so that $f(x)$ is continuous at $x = 1$. Be careful to justify why you know that $f(x)$ will continuous at $x = 1$ for these values of k, l.

Solution:

(a) For any k the expression defines a function $g(x) = \frac{x^2+kx-2}{x^2+5x-5}$ that is continuous everywhere in its domain. This domain contains $x = 1$ since the denominator is not 0 when $x = 1$. Hence,

$$\lim_{x \to 1} \frac{x^2 + kx - 2}{x^2 + 5x - 5} = g(1) = k - 1$$

(b) $f(x)$ is continuous at $x = 1$ if $f(1) = \lim_{x \to 1} f(x)$. This requires

$$l = f(1) = k - 1$$

Hence, $f(x)$ is continuous at $x = 1$ whenever $l = k - 1$.

4. Let $f(x) = \sqrt{2 - x}$.

(a) What is the domain of $f(x)$?

(b) Use the limit definition of the derivative to show that $f'(1) = -1/2$.

(c) Determine the tangent line to the graph $y = f(x)$ at $(1, f(1))$.

Solution:

(a) The domain of $f(x)$ is the collection all real numbers x satisfying $x \leq 2$.

(b) We compute

$$f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} =$$

We have

$$\frac{f(1+h) - f(1)}{h} = \frac{\sqrt{1-h} - 1}{h} = \frac{\sqrt{1-h} - 1}{h} \cdot \frac{\sqrt{1-h} + 1}{\sqrt{1-h} + 1}$$

$$= \frac{1-h-1}{h(\sqrt{1-h}+1)} = \frac{-1}{\sqrt{1-h}+1}$$

Then, this last expression is continuous in h and defined at $h = 0$; hence,

$$f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{-1}{\sqrt{1-h}+1} = \frac{-1}{2}$$

(c) The tangent line at $(1, f(1)) = (1, 1)$ is given by the formula

$$y - 1 = \frac{-1}{2}(x-1) \quad \Longrightarrow \quad y = -\frac{x}{2} + \frac{3}{2}$$

5. (a) Let $h(t)$ be a function with derivative $h'(t)$. Show that

$$\frac{d}{dt}(h(t))^2 = 2h'(t)h(t)$$

(b) Use (a) to compute $\frac{d}{dt}\sqrt{e^t - t^2}$.

Solution:

(a) Using the product rule, we obtain

$$\frac{d}{dt}(h(t))^2 = \frac{d}{dt}h(t) \cdot h(t) = h'(t)h(t) + h(t)h'(t) = 2h'(t)h(t)$$

(b) Let $h(t) = \sqrt{e^t - t^2}$. Then, $(h(t))^2 = e^t - t^2$. Hence, by (a)

$$e^t - 2t = \frac{d}{dt}(h(t))^2 = 2h'(t)(\sqrt{e^t - t^2})$$

$$\Longrightarrow \quad h'(t) = \frac{e^t - 2t}{2\sqrt{e^t - t^2}}$$