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Practice Final: SOLUTION

Disclaimer: This Practice Midterm consists of problems of a similar difficulty as will be on the actual midterm. However, **problems on the actual midterm may or may not be quite different in nature, and may or may not focus on different course material**. However, the actual midterm will have a similar format: one true/false problem, one short-answer problem, three long-answer problems.

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1. True/False (no justification required)

   (a) $\lim_{h\to 0} \frac{e^{1+h} - e}{h} = e$

   (b) If $f''(x) > 0$ then $f(x)$ has a local minimum.

   (c) The limit $\lim_{x\to 0} \frac{\sin(x)}{x}$ does not exist.

   (d) The local linearisation of $f(x) = \sqrt{x}$ at $x = 1$ is $L(x) = 1 + \frac{x}{2}$

   (e) The function $f(x) = |x|$ does not possess an antiderivative.

   (f) If $f'(x) < 1$, for all $0 \le x \le 10$, then $f(x) \le x$, for all $0 \le x \le 10$.

   (g) The formula $\int_0^x f''(x)dx = f'(x) - f'(0)$ holds.

   (h) If $F(x)$ is an antiderivative of $f(x)$ then $F'(x) = f(x)$.

   (i) $\int e^{x^2} dx = \frac{e^{x^2}}{2x} + C$

   (j) The Mean Value Theorem states that if you drive from Waterville, ME, to Portland, ME, at 70 miles per hour, on average, then there is a moment during the drive when you were travelling at exactly 70 miles per hour.

   Solution:

   (a) True: the LHS is $f'(1)$, where $f(x) = e^x$.

   (b) False: $f(x) = e^x$ is a counterexample.

   (c) False: using L'Hopital it can be shown that the limit equals 1

   (d) False: the local linearisation is $1 + \frac{1}{2}(x - 1)$.

   (e) False: the antiderivative is $F(x) = \begin{cases} \frac{1}{2}x^2, x \ge 0, \\ -\frac{1}{2}x^2, x < 0 \end{cases}$

   (f) False: $f(x) = 1 + \frac{1}{2}x$ is a counterexample.

   (g) True: by FFTC

   (h) True: by definition

   (i) False: taking the derivative of RHS does not give e^{x^2} as result.

   (j) True

2. Compute $f'(x)$ for the given function $f(x)$.

(a) $f(x) = \ln(2x+1)\cos(e^x)$

(b) $f(x) = \displaystyle\int_1^{x^2} \sin(t^2)dt$

Solution:

(a) $f'(x) = \frac{2\cos(e^x)}{2x+1} - e^x \ln(2x+1)\sin(e^x)$

(b) $f'(x) = \sin(x^4)2x$

3. Compute the following integral problems.

(a) $\displaystyle\int \frac{1}{x\ln(x)}dx$

(b) $\displaystyle\int_0^1 3x^2\sqrt[3]{2x^3+8}dx$

Solution:

(a) Use u-subsitution: $\displaystyle\int \frac{1}{x\ln(x)}dx = \ln|\ln(x)| + C$

(b) Use u-substitution and FFTC: $\displaystyle\int_0^1 3x^2\sqrt[3]{2x^3+8}dx = \frac{3}{8}\left[\left(2x^3+8\right)^{4/3}\right]_0^1 = \frac{3}{8}\left(10^{4/3} - 8^{4/3}\right)$

4. Let $f(x) = 1 - x$, $g(x) = 1 - x^2$.

(a) Determine the area bounded between the curves $y = f(x)$ and $y = g(x)$.

(b) Compute the equation of the tangent line to $y = g(x)$ at $x = 1/2$.

Solution:

(a) Curves intersect when $x = 0, x = 1$. $A = \int_0^1 g(x) - f(x)dx = \int_0^1 x - x^2 dx = \frac{1}{6}$

(b) $g'(x) = -2x \implies g'(1/2) = -1$. Hence, tangent line is $y - g(1/2) = -1(x - 1/2) \implies y = -x + 5/4$.

5. Consider the sum

$$\sum_{i=1}^{n} \frac{\pi\sin(\pi i/2n)}{2n}$$

(a) Determine a function $f(x)$ and an interval $a \leq x \leq b$ so that the above sum is the corresponding right-hand sum R_n of $f(x)$ on $a \leq x \leq b$.

(b) Using the First Fundamental Theorem of Calculus, compute

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\pi\sin(\pi i/2n)}{2n}$$

Solution:

(a) $f(x) = \sin(x)$, $0 \leq x \leq \pi/2$

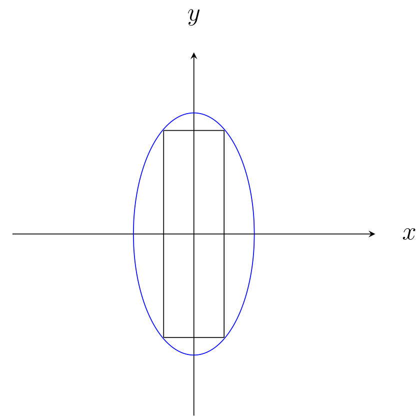
(b) We have $\lim_{n\to\infty} R_n = \int_0^{\pi/2} \sin(x)dx = [-\cos(x)]_0^{\pi/2} = 1$

6. (a) Show that the area A of a rectangle having width $2x$ inscribed inside the ellipse

$$E : \qquad x^2 + \frac{y^2}{4} = 1$$

is

$$A = 4x\sqrt{4 - 4x^2}$$



(b) Explain what's wrong with the following statement: the area of a rectangle having width $2x$ inscribed in E is

$$\int_{-1}^{1} 4x\sqrt{4 - 4x^2}dx$$

(c) Determine the maximal area of a rectangle inscribed in the ellipse E

Solution:

(a) $A = \text{base} \times \text{height}$. We are given that the base is $2x$. The height is $2\sqrt{4 - 4x^2}$: using symmetry, the vertical sides of the rectangle are at $\pm x$, so that the height is twice the distance to the top half of the ellipse. This distance is precisely $\sqrt{4 - 4x^2}$.

(b) The given integral does not depend on x: **all** rectangles would have the same area if this statement were true.

(c) Compute

$$\frac{dA}{dx} = 4\sqrt{4 - 4x^2} - \frac{16x^2}{\sqrt{4 - 4x^2}}$$

We have

$$\frac{dA}{dx} = 0 \quad \Longrightarrow \quad 16x^2 = 4(4 - 4x^2) \quad \Longrightarrow \quad 2x^2 = 1$$

Hence, we have $x = \frac{1}{\sqrt{2}}$ ($x > 0$ because x is a length). This gives $A = \frac{4}{\sqrt{2}}\sqrt{4 - 2} = 4$.
We must check that this is a maximum: note

$$\frac{dA}{dx} = \frac{4(4 - 4x^2) - 16x^2}{\sqrt{4 - 4x^2}} = \frac{16(1 - 2x^2)}{\sqrt{4 - 4x^2}}$$

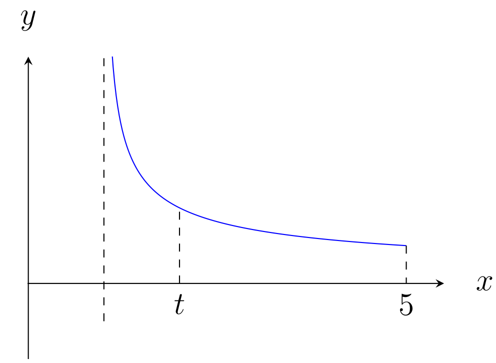
For $0 < x < \frac{1}{\sqrt{2}}$ we see that $\frac{dA}{dx} > 0$; for $x > \frac{1}{\sqrt{2}}$ we see that $\frac{dA}{dx} < 0$. Hence, $x = \frac{1}{\sqrt{2}}$ is a local maximum by the First Derivative Test.

7. Let $f(x) = 1/\sqrt{x - 1}$, defined when $x > 1$.

(a) Let $1 < t < 5$. Compute $A(t) = \displaystyle\int_t^5 f(x)dx$.

(b) Compute $\lim_{t \to 1^+} A(t)$.

(c) True/False: the area bounded below $y = f(x)$, $1 < x \leq 5$, is $\lim_{t \to 1^+} A(t)$. **Justify your answer**.



Solution:

(a) $A(t) = \int_t^5 f(x)dx = [2(x-1)^{1/2}]_t^5 = 2(2 - \sqrt{t-1})$

(b) $\lim_{t \to 1^+} A(t) = 4$

(c) True: the area bounded below $y = f(x)$ is obtained by computing the area below $y = f(x)$, for $t \leq x \leq 5$, and allowing t to get closer and closer, but not equal, to 1 from the right. This is precisely the definition of $\lim_{t \to 1^+} A(t)$.