Colby

Representation Theory
Fall 2019

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# Project & Talk

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**Project Topics:**

1. *The character theory of small groups:*

   In this project you will determine the character tables for all groups $G$ such that $|G| \leq 12$. You are expected to complete the following steps:

   (a) Classify all groups $G$ such that $|G| \leq 12$.

   (b) Use the Orthogonality Relations, and other results in character theory, to determine the character table of each group $G$.

   **Reference:** *Classifying Groups of Small Order* - Gerald Thompson, Adv. in Pure Math http://dx.doi.org/10.4236/apm.2016.62007

2. *The character theory of dihedral groups:*

   In this project you will determine the character tables of the dihedral groups $D_{2n}$. Here, $D_{2n}$ is the group of symmetries of the regular $n$-gon

   $$D_{2n} = \{e, r, r^2, \ldots, r^{n-1}, s, sr, sr^2, \ldots, sr^{n-1}\}$$

   where $r$ is rotation (counter-clockwise) by $2\pi/n$, $s$ is any reflection. Recall that we have the following relations

   $$srs = r^{-1}, \quad r^4 = s^2 = e$$

   (a) Determine the conjugacy classes of $D_{2n}$, when $n = 3, 4, 5, 6, 7, 8$.

   (b) Determine the conjugacy classes of $D_{2n}$, for arbitrary $n$. *Hint: you will have a different case for $n$ odd and $n$ even.*

   (c) You will determine all degree 1 representations via analysis of commutator subgroup as follows: Let $G$ be a group. Denote the **commutator subgroup of** $G$ by $G'$.

   i. Show that $G' \subset G$ is normal.

   ii. Suppose $N \subset G$ is normal subgroup in $G$. Prove: $G/N$ is abelian if and only if $G' \subset N$. In particular, $G/G'$ is abelian.

   iii. Let $\rho : G \to \mathbb{C}^\times$ be a degree 1 representation of $G$. Show that $\rho$ factors *uniquely* through $G/G'$ i.e. there is a *unique* representation $\overline{\rho} : G/G' \to \mathbb{C}^\times$ such that $\rho = \overline{\rho} \circ \pi_{G'}$. Here, $\pi_{G'} : G \to G/G'$ is the quotient homomorphism. *Hint: there is only one choice for $\overline{\rho}$ such that $\overline{\rho} \circ \pi_{G'} = \rho$ - what must the explicit formula for $\overline{\rho}$ be?*
   Deduce that there is a bijection

   $$\{\text{degree 1 reps of } G\}/\text{equiv.} \quad \longleftrightarrow \quad \{\text{irred. reps of } G/G'\}/\text{equiv.}$$

iv. Determine the commutator subgroup $D'_{2n}$ of $D_{2n}$ Show that

$$D_{2n}/D'_{2n} \simeq \begin{cases} \mathbb{Z}/2\mathbb{Z}, \ n \text{ odd}, \\ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \ n \text{ even} \end{cases}$$

v. Determine the degree 1 representations of $D_{2n}$.

(d) Determine degree 2 unitary representations. *Hint: this is way easier than the degree 1 case! Modify unitary representations we have seen in class or feel free to come and ask.*

(e) Using these representations, compute the character table.

**Reference:** *Dihedral Groups* - Keith Conrad,

`https://kconrad.math.uconn.edu/blurbs/grouptheory/dihedral.pdf`

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**Project Guidelines:**

1. You should submit a (hard copy) 8-12 page report, written in LaTeX (single-spaced), outlining your investigation. The preliminary submission deadline is **Friday, April 26, 4pm**. Projects will be provided with some feedback and returned Monday, April 29, so that suggestions/improvements can be incorporated. The final submission deadline will be **Monday, May 6, 1pm.**

2. Your report should include a 'summary' of solutions to the problems outlined above. You are free to interpret 'summary' as you please. In particular, you need not include all details for all steps: a good example can tell a thousand words :)

3. You are allowed to consult any materials that you wish (e.g. textbooks, online), but all resources used should be properly cited. You are allowed to collaborate on your projects but you should cite who you work with whenever appropriate. You are very welcome to ask me (or anyone else) for guidance.

4. You will be required to give a 15 minute talk about your project. More details to follow.