Representation Theory
Spring 2019
Contact: `gwmelvin@colby.edu`

Midterm Review

Definitions

You should know the following definitions:

1. Basic linear algebra: **span**, **linear independence**, **basis**, **matrix of a linear map with respect to a basis**, **change-of-basis matrix**.
2. Definition of **eigenvalue**, **eigenvector**, **λ-eigenspace**, **characteristic polynomial**.
3. You should know what it means for a linear map to be **diagonalisable**.
4. Definition of **inner product space**, **orthogonal set**, **orthonormal set**, **orthogonal complement**.
5. Definition of the **adjoint** of a linear map.
6. Definition of a **unitary**, **self-adjoint** linear map.
7. Definition of **representation**, **subrepresentation**, **degree**.
8. Definition of **direct sum representation**.
9. What it means for a representation to be **irreducible**, **(in)decomposable**, **completely reducible**.
10. Definition of a **unitary representation**.
11. Definition of **G-(iso)morphism**.
12. Definition of the vector spaces $\mathrm{Hom}_G(\varphi, \rho), \mathrm{End}_G(\rho)$.
13. What it means for two representations to be **equivalent**.
14. Definition of the **group algebra**.
15. Definition of **matrix elements/coefficients** of a matrix representation.
16. Definition of the **character** of a representation.
17. Definition of an **irreducible character**.
18. Definition of **class function**.

Results

You should know the following theorems, propositions etc. Results with a $*$ are theorems, propositions etc whose proof you should understand and be able to reproduce.

1. You should know the relationship between a linear map T and the matrix associated to T with respect to a basis B.

2. You should know the relationship between diagonalisability and eigenthings.

3. You should know the fundamental relationship between a linear map and its adjoint.

4. You should know that a matrix is unitary if and only if its columns are orthonormal.

5. You should know that a subrepresentation is a representation in its own right (i.e. what $\rho_{|U}$ means).

6. You should know the relationships between: subpresentations and block upper-triangular matrices; direct sum representations and block diagonal matrices.

7. You should know that if a representation is equivalent to a direct sum then there exists subrepresentations, with respect to which, the original representation is the internal direct sum.

8. You should know that equivalent representations admit bases so that the resulting matrix representations are identical. **You do not need to know the proof of this statement.**

9. You should know that a degree 1 subrepresentation of ρ corresponds to finding a common eigenvector for ρ_g, $g \in G$.

10. $*$ You should know the Unitary Version of Maschke's Theorem.

11. You should know that any representation is a unitary representation with respect to some inner product.

12. You should know the statement of Maschke's Theorem and how it follows from the Unitary Version of Maschke's Theorem.

13. $*$ You should know that the kernel/image of a G-morphism is a subrepresentation.

14. $*$ You should know Schur's Lemma.

15. $*$ You should know that irreducible representations of finite abelian groups have degree 1.

16. You should know that any representation ρ of a finite abelian group may be simultaneously diagonalised i.e. there is a basis B, with respect to which, the matrices $[\rho_g]_B$, $g \in G$, are diagonal.

17. You should know that the group algebra is an inner product space; in particular, you should know what the inner product is.

18. You should know the Schur Orthogonality Relations.

19. You should know that there are only finitely many equivalence classes of irreducible representations.

20. $*$ You should know that equivalent representations have equal characters.

21. $*$ You should know that characters are class functions.

22. You should know that the space of all class functions has dimension equal to the number of conjugacy classes in G.

23. $*$ You should know that the set of all irreducible characters is orthonormal.

24. ~~You should know that the number of conjugacy classes in G is equal to the number of irreducible characters.~~

25. You should know that the character of a direct sum is the sum of the characters of the summands.

Computations & Examples

You should feel comfortable with the following examples and know how to perform the following computations:

1. How to find the matrix of a linear map with respect to some basis.
2. How to find eigenthings.
3. How to diagonalise a linear map/matrix.
4. How to determine the matrix (with respect to an orthonormal basis) of the adjoint.
5. How to show a subspace is a subrepresentation.
6. How to identify a direct sum representation from a matrix representation.
7. How to find a degree 1 subrepresentation.
8. How to show that a degree 2 representation is irreducible.
9. How to check that a linear map is a G-morphism.
10. How to compute the character of a given representation of small groups (e.g. S_3, D_8) and cyclic groups.
11. How to use the character to check if a representation is irreducible.
12. The trivial representation.
13. The standard representation of S_n.
14. The two degree 2 representations of D_8 given in class.