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Midterm Practice Problems

The problems on the midterm will not be as long/difficult as the following problems. These problems are intended to give you practice proving and computing things.

Throughout these problems G denotes a finite group, V a finite dimensional vector space over $\mathbb{C}$.

1. While the statement of this problem holds, the proof is a bit harder than anticipated so feel free to disregard... Sorry! Let (ρ, V) be a representation of G. Show that

$$\ker \chi_\rho \stackrel{def}{=} \{g \in G \mid \chi_\rho(g) = \deg \rho\} \subset G$$

is a normal subgroup.

2. Let $N \subset G$ be a normal subgroup and let $H = G/N$. Denote the quotient homomorphism

$$\pi_N : G \to H \ , \ g \mapsto gN$$

   (a) Let (φ, W) be a representation of H. Show that $\rho = \varphi \circ \pi_N$ is a representation of G. *This should be very straighforward!*

   (b) Let $G = S_3$, $N = \{e, (123), (132)\}$. Show that N is normal in G.

   (c) Show that $G/N \simeq \mathbb{Z}/2\mathbb{Z}$.

   (d) Let $\varphi' : \mathbb{Z}/2\mathbb{Z} \to \mathrm{GL}(\mathbb{C})$ be the (unique) nontrivial irreducible representation. Compute the character $\chi_{\rho'}$ of $\rho' = \varphi' \circ \pi_N$.

   (e) Explain in two different ways why ρ' is irreducible. (*One way requires one sentence, the other involves characters*)

   (f) Now let φ be the standard representation of $\mathbb{Z}/2\mathbb{Z}$. Is the representation $\rho = \varphi \circ \pi_N$ irreducible? (*Hint: characters might help!*)

3. Let ρ be a representation with the property that $\chi_\rho(g) = m_g \in \mathbb{Z}_{\geq 0}$, for every $g \in G$.

   (a) Show that the trivial representation is an irreducible constituent of ρ.

   (b) Show that $|G|$ divides $\sum_{g \in G} \chi_\rho(g)$.

4. Let X be a nonempty set.

   - A homomorphism $\alpha : G \to \mathrm{Perm}(X)$ is called an **action of** G **on** X (we also say that G **acts on** X). Here $\mathrm{Perm}(X) = \{f : X \to X \mid f \text{ bijective}\}$ is the group of bijections on X. In particular, a representation (ρ, V) of G is an action of G on V.
   - We define $\mathbb{C}[X] = \{f : X \to \mathbb{C}\}$, the set of all $\mathbb{C}$-valued functions on X. In particular, when $X = G$ we recover the group algebra. It can be shown that $\mathbb{C}[X]$ is a vector space over $\mathbb{C}$.

Assume for the remainder of this problem that X is finite. In this case, $\mathbb{C}[X]$ has dimension $|X|$: a basis is given by $\{e_x \mid x \in X\}$, where

$$e_x(y) = \begin{cases} 1, & x = y \\ 0, & x \neq y \end{cases}$$

The proof is exactly the same as the proof used to determine the dimension of $\mathbb{C}[G]$.

(a) Let $\alpha : G \to \mathrm{Perm}(X)$ be a group action. Show that

$$\rho_\alpha : G \to \mathrm{GL}(\mathbb{C}[X]) \ , \ g \mapsto (\rho_\alpha)_g$$

where, for any $f \in \mathbb{C}[X]$,

$$((\rho_\alpha)_g(f))(x) = f(\alpha(g^{-1})(x))$$

defines a representation of G.

(b) Let $g \in G, x \in X$. Show that $(\rho_\alpha)_g(e_x) = e_{\alpha(g)(x)}$.

(c) Let $v = \sum_{x \in X} e_x$. Show that $\mathrm{span}(v) \subset \mathbb{C}[X]$ is a subrepresentation. Hence, deduce that $\mathbb{C}[X]$ is not irreducible.

5. Consider the representation of D_8

$$\begin{aligned} \psi : \ D_8 &\to \mathrm{GL}_2(\mathbb{C}) \\ r &\mapsto \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\ s &\mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \end{aligned}$$

Consider the square having vertices $(\pm 1, 0), (0, \pm 1)$. Let L_x, L_y denote the x- and y-axis respectively. Define $X = \{L_x, L_y\}$ and write $e_x = e_{L_x}, e_y = e_{L_y} \in \mathbb{C}[X]$. Set $B = \{e_x, e_y\} \subset \mathbb{C}[X]$.

Observe that ψ_r, ψ_s preserve X:

$$\psi_r(L_x) = L_y, \ \psi_r(L_y) = L_x \quad \text{and} \quad \psi_s(L_x) = L_x, \ \psi_s(L_y).$$

Hence, ψ induces an action α of D_8 on X: by construction, $\alpha(r), \alpha(s)$ are the permutations of X given above and $\alpha(s^i r^j) = \alpha(s)^i \alpha(r)^j$ is the composition of the corresponding permutations given above. That this is well-defined (i.e. α is a homomorphism) follows because α is determined by the homomorphism ψ and the fact that ψ_g is **linear**.

You can visualise this action by thinking about how ψ_g, for $g \in G$, permutes the x- and y-axes.

Denote the resulting representation on $\mathbb{C}[X]$ by ψ_α.

(a) Using (b) in the previous Problem, show that

$$[(\psi_\alpha)_r]_B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad [(\psi_\alpha)_s]_B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(b) Compute the character of ψ_α.

(c) Show that ψ_α is not irreducible.

(d) Show that the trivial representation $\mathrm{triv} : D_8 \to \mathrm{GL}(\mathbb{C})$ is an irreducible constituent of ψ_α. (*Hint: compute* χ_{triv} *and determine the multiplicity of* triv *in* ψ_α)