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March 8: Maschke's Theorem

Convention: Unless otherwise specified, G will always denote a finite group, V a finite dimensional vector space over $\mathbb{C}$.

Lemma (6.1): *Let V be a finite dimensional vector space over $\mathbb{C}$. Then, V admits an inner product.*

Proof: Choose a basis $B \subseteq V$. Define

$$\langle , \rangle : V \times V \to \mathbb{C}$$

via the formula

$$\langle u, v\rangle = \langle [u]_B, [v]_B\rangle_s$$

where $\langle , \rangle_s$ denotes the standard inner product on $\mathbb{C}^k$ ($k = \dim V$). Let's check that this defines an inner product on V: let $u, v, w \in V$, $\lambda, \mu \in \mathbb{C}$.

1. $$\begin{aligned}\langle \lambda u + \mu v, w\rangle &= \langle [\lambda u + \mu v]_B, [w]_B\rangle_s \\ &= \langle \lambda [u]_B + \mu [v]_B, [w]_B\rangle_s \\ &= \lambda\langle [u]_B, [w]_B\rangle_s + \mu\langle [v]_B, [w]_B\rangle_s \\ &= \lambda\langle u, w\rangle + \mu\langle v, w\rangle\end{aligned}$$

2. $\langle u, v\rangle = \langle [u]_B, [v]_B\rangle_s = \overline{\langle [v]_B, [u]_B\rangle_s} = \overline{\langle v, u\rangle}$

3. $\langle u, u\rangle = \langle [u]_B, [u]_B\rangle_s \geq 0$ and $\langle u, u\rangle = 0$ if and only if $\langle [u]_B, [u]_B\rangle_s = 0$ if and only if $[u]_B = \underline{0}$ if and only if $u = 0_V$.

QED

Proposition (6.2): *Let (ρ, V) be a nonzero representation of a finite group G. Then, there exists an inner product on V with respect to which ρ is unitary.*

Proof: Let $\langle , \rangle$ be an inner product on V, which exists by Lemma 6.1. Define

$$[,] : V \times V \to \mathbb{C}\ ,\ (u, v) \mapsto [u, v] = \sum_{g \in G} \langle \rho_g(u), \rho_g(v)\rangle$$

Then, $[,]$ is an inner product on V: the proof of sesquilinearity is similar to the proof of Lemma 6.1. Also, we have

$$[u, u] = \sum_{g \in G} \langle \rho_g(u), \rho_g(u)\rangle \geq 0$$

and $[u, u] = 0$ if and only if each term of the above sum equals 0. In particular, $0 = \langle \rho_e(u), \rho_e(u)\rangle = \langle u, u\rangle$ if and only if $u = 0_V$.

Let's show that $[,]$ is an inner product for which ρ_g is unitary, for every $g \in G$. Indeed, for $u, v \in V$, $g \in G$,

$$[\rho_g(u), \rho_g(v)] = \sum_{h \in G} \langle \rho_h(\rho_g(u)), \rho_h(\rho_g(v)) \rangle = \sum_{h \in G} \langle \rho_{hg}(u), \rho_{hg}(v) \rangle$$

Recall that $r_g : G \to G$, $h \mapsto hg$ is a bijection on G. Hence, the right-hand sum above is a permutation of $\sum_{h \in G} \langle \rho_h(u), \rho_h(v) \rangle$. Hence, ρ_g is unitary, for every $g \in G$.

QED

Corollary (6.3): (Maschke's Theorem)

Let (ρ, V) be a nonzero representation of the finite group G. Then, ρ is completely reducible.

Proof: By Proposition 6.2, we can choose an inner product on V with respect to which ρ is unitary. The result follows from Corollary 5.5.

QED

Section III: G-morphisms

Definition (6.4): Let (ρ, V), (φ, W) be representations of G. A G**-morphism from** ρ **to** φ (or **from** V **to** W) is a linear map $T : V \to W$ such that

$$T \circ \rho_g = \varphi_g \circ T, \qquad \text{for every } g \in G$$

This means that, for every $v \in V$, $g \in G$,

$$T(\rho_g(v)) = \varphi_g(T(v)).$$