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March 6: Maschke's Theorem

Convention: Unless otherwise specified, G will always denote a finite group, V a finite dimensional vector space over $\mathbb{C}$.

Section II: Maschke's Theorem

In this section we will prove one of the fundamental results in the representation theory of finite groups - *Maschke's Theorem*. This theorem states that, for the representation theory of finite groups (defined over $\mathbb{C}$), irreducibility and indecomposability coincide.

Definition (5.1): Let $(V, \langle , \rangle)$ be an inner product space, $\rho : G \to \mathrm{GL}(V)$ a representation of G. We say that ρ is **unitary** if ρ_g is unitary, for every $g \in G$. This means, for every $g \in G$, $u, v \in V$, $\langle \rho_g(u), \rho_g(v)\rangle = \langle u, v\rangle$.

Example (5.2):

1. Any degree 1 representation $\rho : G \to \mathrm{GL}(\mathbb{C})$ is unitary: here $\mathbb{C}$ is equipped with the standard inner product.

2. The representation
$$\begin{aligned}\rho : \quad D_8 &\to GL_2(\mathbb{C})\\ r &\mapsto \begin{bmatrix} i & 0\\ 0 & -i\end{bmatrix}\\ s &\mapsto \begin{bmatrix} 0 & 1\\ 1 & 0\end{bmatrix}\end{aligned}$$
is unitary. Indeed,
$$\begin{aligned}\left\langle \rho_r\left(\begin{bmatrix} a_1\\ a_2\end{bmatrix}\right), \rho_r\left(\begin{bmatrix} b_1\\ b_2\end{bmatrix}\right)\right\rangle &= \left\langle \begin{bmatrix} ia_1\\ -ia_2\end{bmatrix}, \begin{bmatrix} ib_1\\ -ib_2\end{bmatrix}\right\rangle\\ &= ia_1\overline{(ib_1)} + (-ia_2)\overline{(-ib_2)}\\ &= a_1\bar{b}_1 + a_2\bar{b}_2\\ &= \left\langle \begin{bmatrix} a_1\\ a_2\end{bmatrix}, \begin{bmatrix} b_1\\ b_2\end{bmatrix}\right\rangle\end{aligned}$$
A similar calculation shows ρ_s is unitary.

3. The representation
$$\rho : \mathbb{Z}/3\mathbb{Z} \to \mathrm{GL}_2(\mathbb{C}) \ , \ \bar{j} \mapsto \begin{bmatrix} 0 & -1\\ 1 & -1\end{bmatrix}^j$$
is not unitary: we have $\rho_{\bar{1}}\left(\begin{bmatrix} 1\\ 1\end{bmatrix}\right) = \begin{bmatrix} -1\\ 0\end{bmatrix}$. Since
$$\left|\left|\rho_{\bar{1}}\left(\begin{bmatrix} 1\\ 1\end{bmatrix}\right)\right|\right|^2 = 1 \neq 2 = \left|\left|\begin{bmatrix} 1\\ 1\end{bmatrix}\right|\right|^2$$
we conclude that $\rho_{\bar{1}}$ is not unitary.

Proposition (5.3): *Let (ρ, V) be a nonzero unitary representation. Then, either ρ is irreducible or decomposable.*

Proof: Suppose that ρ is not irreducible. Then, there exists a nonzero proper subrepresentation $W \subseteq V$.

Claim: $W^{\perp}$ is a subrepresentation.

Indeed, let $g \in G$, $w \in W^{\perp}$. Then, for any $u \in W$,

$$\begin{aligned}\langle u, \rho_g(w)\rangle &= \langle \rho_{g^{-1}}(u), \rho_{g^{-1}}(\rho_g(w))\rangle \\ &= \langle \rho_{g^{-1}}(u), w\rangle \\ &= 0, \qquad \text{since } \rho_{g^{-1}}(u) \in W \text{ and } w \in W^{\perp}.\end{aligned}$$

Hence, $\rho_g(w) \in W^{\perp}$. Moreover, $\rho \simeq \rho_{|_W} \oplus \rho_{|_{W^{\perp}}}$: we have a G-isomorphism

$$T : W \times W^{\perp} \to V \ , \ (w, u) \mapsto w + u$$

- T is a linear isomorphism by Homework.

- we have

$$T(\rho_{|_W} \oplus \rho_{|_{W^{\perp}}}(w, u)) = T(\rho_g(w), \rho_g(u)) = \rho_g(w) + \rho_g(u) = \rho_g(w+u) = \rho_g(T(w, u))$$

Remark (5.3): If $U, W \subseteq V$ are subrepresentations such that $V = U \oplus W$ (internal direct sum) then $\rho \simeq \rho_{|_U} \oplus \rho_{|_W}$

Corollary (5.5): (Maschke's Theorem for Unitary Representations)
Every nonzero unitary representation (ρ, V) is completely reducible.

Proof: Proceed by induction on $k = \dim V$.

$(k = 1)$: any degree 1 representation is irreducible.

Assume the result for all $1 \leq k < r$. Let $k = \dim V = r$.

- If (ρ, V) is irreducible then we are done.

- Else, by Proposition 5.3, $\rho \simeq \rho_1 \oplus \rho_2$. Hence, by induction, ρ_1 and ρ_2 are completely reducible

$$\rho_1 \simeq \varphi_1 \oplus \cdots \oplus \varphi_s, \qquad \rho_2 \simeq \psi_1 \oplus \cdots \oplus \psi_t$$

  with each φ_i, ψ_j irreducible. Then,

$$\rho \simeq \varphi_1 \oplus \cdots \oplus \varphi_s \oplus \psi_1 \oplus \cdots \oplus \psi_t$$