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March 4: Decomposability

Convention: Unless otherwise specified, G will always denote a finite group, V a finite dimensional vector space over $\mathbb{C}$.

Example (4.1): Let $G = \mathbb{Z}/4\mathbb{Z}$. There is a homomorphism

$$f : \mathbb{Z}/4\mathbb{Z} \to D_8 \ , \ \bar{j} \mapsto r^j$$

that gives rise to a degree 2 representation of $\mathbb{Z}/4\mathbb{Z}$:

$$\rho : \mathbb{Z}/4\mathbb{Z} \to D_8 \to \mathrm{GL}_2(\mathbb{C}) \ , \ \bar{j} \mapsto \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^j$$

Observe:

- $U = \mathrm{span}\left(\begin{bmatrix} 1 \\ i \end{bmatrix}\right)$ is a subrepresentation

$$\rho(\bar{j})\left(\begin{bmatrix} 1 \\ i \end{bmatrix}\right) = (-i)^j \begin{bmatrix} 1 \\ i \end{bmatrix}$$

- $W = \mathrm{span}\left(\begin{bmatrix} 1 \\ -i \end{bmatrix}\right)$ is a subrepresentation

$$\rho(\bar{j})\left(\begin{bmatrix} 1 \\ -i \end{bmatrix}\right) = i^j \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

Define the degree 1 representations of $\mathbb{Z}/4\mathbb{Z}$:

$$\rho_1 : \mathbb{Z}/4\mathbb{Z} \to \mathrm{GL}(\mathbb{C}) \ , \ \bar{j} \mapsto (-i)^j$$

$$\rho_2 : \mathbb{Z}/4\mathbb{Z} \to \mathrm{GL}(\mathbb{C}) \ , \ \bar{j} \mapsto i^j$$

Then, $\rho \simeq \rho_1 \oplus \rho_2$ via the G-isomorphism

$$T : \mathbb{C} \times \mathbb{C} \to \mathbb{C}^2 \ , \ (a,b) \mapsto \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = a \begin{bmatrix} 1 \\ i \end{bmatrix} + b \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

Indeed, we have

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^j \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} (-i)^j & i^j \\ i \cdot (-i)^j & (-i) \cdot i^j \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = (-i)^j a \begin{bmatrix} 1 \\ i \end{bmatrix} + i^j b \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

That is,

$$\rho(\bar{j})(T(a,b)) = T((\rho_1 \oplus \rho_2)(\bar{j})(a,b))$$

verifying that T is a G-isomorphism.

Remark (4.2): If $\rho \simeq \rho_1 \oplus \rho_2$ then there exists a basis $B \subseteq V$ such that

$$[\rho_g]_B = \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}, \qquad \text{for every } g \in G,$$

where the $*$ in the top left is $\dim V_1 \times \dim V_1$, and the $*$ in the bottom right is $\dim V_2 \times \dim V_2$. Namely, choose bases $B_1 \subseteq V_1$, $B_2 \subseteq V_2$ and let $T : V \to V_1 \times V_2$ be a G-isomorphism. Then, $B = T^{-1}(B_1 \times \{0\}) \cup T^{-1}(\{0\} \times B_2)$ is a basis with this property.

Conversely, if $B = (v_1, \dots, v_k) \subseteq V$ is a basis such that

$$[\rho_g]_B = \begin{bmatrix} \overset{m \times m}{*} & 0 \\ 0 & \overset{n \times n}{*} \end{bmatrix}, \qquad \text{for every } g \in G,$$

with $m + n = k$, then $\rho \simeq \rho_{|U} \oplus \rho_{||W}$, where

$$U = \operatorname{span}(v_1, \dots, v_m), \qquad W = \operatorname{span}(v_{m+1}, \dots, v_{m+n}).$$

Definition (4.3): Let (ρ, V) be a nonzero representation of G. We say that ρ is **decomposable** if $\rho \simeq \rho_1 \oplus \rho_2$, with each of ρ_1, ρ_2 nonzero. Otherwise, we say that ρ is **indecomposable**.

Remark (4.4):

- Irreducible $\implies$ indecomposable.
- Completely reducible $\implies$ decomposable.

Interesting Example (4.5):

1. Let $G = \mathbb{Z}$. Then
$$\rho : G \to \mathrm{GL}_2(\mathbb{C}) \ , \ n \mapsto \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$$
is a representation of $\mathbb{Z}$. However, $\operatorname{span}(e_1)$ is the only proper subrepresentation (i.e. one eigenvalue $\lambda = 1$ with 1-dim eigenspace). Hence, ρ is indecomposable but not irreducible.

2. Let $G = \mathbb{Z}/p\mathbb{Z}$. Define
$$\rho : G \to \mathrm{GL}_2(\mathbb{F}_p) \ , \ \bar{j} \mapsto \begin{bmatrix} 1 & \bar{j} \\ 0 & 1 \end{bmatrix}$$
Similarly, $\operatorname{span}(e_1)$ is the only subrepresentation. Hence, ρ is indecomposable but not irreducible.