Contact: `gwmelvin@colby.edu`

March 20: Towards Schur Orthogonality

Convention: Unless otherwise specified, G will always denote a finite group, V a finite dimensional vector space over $\mathbb{C}$.

Lemma (11.1): *Let $(V, \langle , \rangle_V)$, $(W, \langle , \rangle_W)$ be inner product spaces. Let (φ, V), (ρ, W) be unitary representations. Let $B \subseteq V$, $C \subseteq W$ be orthonormal bases, and $\{\varphi_{ij}\}, \{\rho_{rs}\} \subseteq \mathbb{C}[G]$ be the corresponding matrix elements. Then,*

$$[(L_{ri}^{B,C})^{\#}]_B^C = [\langle \varphi_{ij}, \rho_{rs}\rangle]_{sj}$$

i.e. the s, j-entry of $[(L_{ri}^{B,C})^{\#}]_B^C$ is $\langle \varphi_{ij}, \rho_{rs}\rangle$.

Proof: Let $B = (v_1, \dots, v_k)$. To determine the j^{th} column of $[(L_{ri}^{B,C})^{\#}]_B^C$ we need to evaluate the following:

$$\begin{aligned}\left(L_{ri}^{B,C}\right)^{\#}(v_j) &= \frac{1}{|G|}\sum_{g\in G} \rho_{g^{-1}} L_{ri}^{B,C}\varphi_g(v_j)\\ &= \frac{1}{|G|}\sum_{g\in G} \rho_{g^{-1}}\left(L_{ri}^{B,C}\left(\varphi_g(v_j)\right)\right)\end{aligned}$$

Since C is orthonormal,

$$[(L_{ri}^{B,C})^{\#}(v_j)]_C = \begin{bmatrix}\langle (L_{ri}^{B,C})^{\#}(v_j), w_1\rangle_W\\ \vdots \\ \langle (L_{ri}^{B,C})^{\#}(v_j), w_l\rangle_W\end{bmatrix}$$

For $s = 1, \dots, l$,

$$\begin{aligned}\langle (L_{ri}^{B,C})^{\#}(v_j), w_s\rangle_W &= \frac{1}{|G|}\sum_{g\in G}\left\langle \rho_{g^{-1}}\left(L_{ri}^{B,C}\left(\varphi_g(v_j)\right)\right), w_s\right\rangle_W\\ &= \frac{1}{|G|}\sum_{g\in G}\left\langle L_{ri}^{B,C}\left(\varphi_g(v_j)\right), \rho_g(w_s)\right\rangle_W, \quad \text{since } \rho \text{ unitary.}\end{aligned}$$

By construction, if $[\varphi_g(v_j)]_B = \begin{bmatrix}a_1\\ \vdots\\ a_k\end{bmatrix} = \begin{bmatrix}\langle \varphi_g(v_j), v_1\rangle_V\\ \vdots \\ \langle \varphi_g(v_j), v_k\rangle_V\end{bmatrix}$, then

$$L_{ri}^{B,C}(\varphi_g(v_j)) = a_i w_r = \langle \varphi_g(v_j), v_i\rangle_V w_r$$

Hence,

$$\begin{aligned}\langle (L_{ri}^{B,C})^{\#}(v_j), w_s\rangle_W &= \frac{1}{|G|}\sum_{g\in G}\langle \langle \varphi_g(v_j), v_i\rangle_V w_r, \rho_g(w_s)\rangle_W\\ &= \frac{1}{|G|}\sum_{g\in G}\langle \varphi_g(v_j), v_i\rangle_V \langle w_r, \rho_g(w_s)\rangle_W\\ &= \frac{1}{|G|}\sum_{g\in G}\langle \varphi_g(v_j), v_i\rangle_V \overline{\langle \rho_g(w_s), w_r\rangle_W}\end{aligned}$$

Observe, if $L : V \to V$ is linear then the i, j entry of $[L]_B$ is $\langle L(v_j), v_i \rangle_V$. Similarly, if $K : W \to W$ is linear then the i, j-entry of $[K]_C$ is $\langle K(w_j), w_i \rangle_W$. Hence,

$$\phi_{ij}(g) = \langle \varphi_g(v_j), v_i \rangle_V, \qquad \rho_{rs}(g) = \langle \rho_g(w_s), w_r \rangle_W$$

so that

$$\langle (L_{ri}^{B,C})^{\#}(v_j), w_s \rangle_W = \frac{1}{|G|} \sum_{g \in G} \varphi_{ij}(g) \overline{\rho_{rs}(g)} = \langle \varphi_{ij}, \rho_{rs} \rangle$$

In particular, the s, j-entry of $[(L_{ri}^{B,C})^{\#}]_B^C$ is $\langle \varphi_{ij}, \rho_{rs} \rangle$.

QED

Example (10.1): Let $G = D_8$ and consider the representation

$$\begin{array}{rccc} \rho: & D_8 & \to & GL_2(\mathbb{C}) \\ & r & \mapsto & \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \\ \\ & s & \mapsto & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array}$$

Then, ρ is irreducible and unitary (with respect to the standard inner product on $\mathbb{C}^2$). We have

$$e \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad s \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
$$r \mapsto \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \qquad sr \mapsto \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$
$$r^2 \mapsto \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad sr^2 \mapsto \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$
$$r^3 \mapsto \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}, \qquad sr^3 \mapsto \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$$

Then,

$$\rho_{11} - \text{ picks out the } 1,1\text{-entry}$$
$$\rho_{12} - \text{ picks out the } 1,2\text{-entry}$$

We compute

$$\langle \rho_{11}, \rho_{12} \rangle = \frac{1}{8} (1 \cdot 0 + i \cdot 0 + (-1) \cdot 0 + (-i) \cdot 0 + 0 \cdot 1 + 0 \cdot i + 0 \cdot (-1) + 0 \cdot (-i)) = 0$$

and

$$\langle \rho_{11}, \rho_{11} \rangle = \frac{1}{8} \left(|1|^2 + |i|^2 + |-1|^2 + |-i|^2 + 0^2 + 0^2 + 0^2 + 0^2 \right) = \frac{1}{2}$$

Proposition (10.2): *Let $\varphi : G \to \mathrm{GL}(V)$, $\rho : G \to \mathrm{GL}(W)$ be representations of G, $T : V \to W$ a linear map (not necessarily a G-morphism). Define*

$$T^{\#} = \frac{1}{|G|} \sum_{g \in G} \rho_{g^{-1}} T \varphi_g : V \to W$$

1. *$T^{\#}$ is a G-morphism.*
2. *If $T \in \mathrm{Hom}_G(\varphi, \rho)$ then $T^{\#} = T$.*
3. *The function $P : \mathrm{Hom}(V, W) \to \mathrm{Hom}(V, W)$, $T \mapsto T^{\#}$ is linear and $\mathrm{im}\, P = \mathrm{Hom}_G(V, W)$.*

Proof:

1. Since $\rho_{g^{-1}} T \varphi_g : V \to W$, and $T^{\#}$ is a linear combination of these linear maps, $T^{\#}$ is linear.

   Let $x \in G$. Then,

$$\begin{aligned}
\rho_x T^{\#} &= \frac{1}{|G|} \sum_{g \in G} \rho_x \rho_{g^{-1}} T \varphi_g \\
&= \frac{1}{|G|} \sum_{g \in G} \rho_{xg^{-1}} T \varphi_{gx^{-1}x} \\
&= \frac{1}{|G|} \sum_{g \in G} \rho_{xg^{-1}} T \varphi_{gx^{-1}} \varphi_x \\
&= \left(\frac{1}{|G|} \sum_{g \in G} \rho_{xg^{-1}} T \varphi_{gx^{-1}} \right) \varphi_x \\
&= T^{\#} \varphi_x
\end{aligned}$$

   Here we use that $\frac{1}{|G|} \sum_{g \in G} \rho_{xg^{-1}} T \varphi_{gx^{-1}}$ is a rearrangement of the sum defining $T^{\#}$.

2. Suppose $T \in \mathrm{Hom}_G(\varphi, \rho)$. Then,

$$\begin{aligned}
T^{\#} &= \frac{1}{|G|} \sum_{g \in G} \rho_{g^{-1}} T \rho_g \\
&= \frac{1}{|G|} \sum_{g \in G} T \rho_{g^{-1}} \rho_g \\
&= \frac{1}{|G|} \sum_{g \in G} T = T
\end{aligned}$$

3. For $T, S \in \mathrm{Hom}(V, W)$,

$$(T+S)^{\#} = \frac{1}{|G|} \sum_{g \in G} \rho_{g^{-1}} (T+S) \varphi_g = \frac{1}{|G|} \sum_{g \in G} (\rho_{g^{-1}} T \varphi_g + \rho_{g^{-1}} S \varphi_g) = T^{\#} + S^{\#}$$

   For $c \in \mathbb{C}$, $T \in \mathrm{Hom}(V, W)$,

$$(cT)^{\#} = \frac{1}{|G|} \sum_{g \in G} \rho_{g^{-1}} (cT) \varphi_g = c \left(\frac{1}{|G|} \sum_{g \in G} \rho_{g^{-1}} T \varphi_g \right) = cT^{\#}$$

   Hence, P is linear. By (1) and (2), we have $\mathrm{im}\, P = \mathrm{Hom}_G(V, W)$.

QED

Proposition (10.3): *Let $\varphi : G \to \mathrm{GL}(V)$, $\rho : G \to \mathrm{GL}(W)$ be irreducible representations, $T : V \to W$ a linear map. Then,*

1. *If φ and ρ are inequivalent then $T^{\#} = 0$.*
2. *IF $\varphi = \rho$ then $T^{\#} = \frac{\mathrm{tr}(T)}{\deg \varphi} \mathrm{id}_V$.*

Proof:

1. By 10.2, (1), $T^{\#} \in \mathrm{Hom}_G(\varphi, \rho)$. Now, Schur's Lemma gives $T^{\#} = 0$ since φ, ρ are inequivalent representations.

2. If $\varphi = \rho$ then $T = \lambda \mathrm{id}_V$, for some $\lambda \in \mathbb{C}$, by Schur's Lemma and 10.2, (1). Hence, we have
$$\mathrm{tr}(T^{\#}) = \mathrm{tr}(\lambda \mathbb{I}_{\deg \varphi}) = \lambda \deg \varphi$$
Also,
$$\begin{aligned}\mathrm{tr}(T^{\#}) &= \mathrm{tr}\left(\frac{1}{|G|}\sum_{g\in G} \varphi_{g^{-1}} T \varphi_g\right) \\ &= \frac{1}{|G|}\sum_{g\in G} \mathrm{tr}(\varphi_{g^{-1}} T \varphi_g) \\ &= \frac{1}{|G|}\sum_{g\in G} \mathrm{tr}(T \varphi_g \varphi_{g^{-1}}) \\ &= \frac{1}{|G|}\sum_{g\in G} \mathrm{tr}(T) = \mathrm{tr}(T)\end{aligned}$$
Hence,
$$\lambda \deg \varphi = \mathrm{tr}(T) \quad \implies \quad T^{\#} = \frac{\mathrm{tr}(T)}{\deg \varphi}\mathrm{id}_V$$

Remark (10.4): Let $B = (v_1, \ldots, v_k) \subseteq V$, $C = (w_1, \ldots, w_l) \subseteq W$ be bases. Define the linear map $L_{ij}^{B,C} : V \to W$ to be the unique linear map satisfying
$$[L_{ij}^{B,C}]_B^C = E_{ij}$$
where E_{ij} is the $l \times k$ matrix with a 1 in the ij-entry and 0s elsewhere. In particular, for $v \in V$, with $[v]_B = \begin{bmatrix} a_1 \\ \vdots \\ a_k \end{bmatrix}$, we have $L_{ij}(v) = a_j w_i$.
Since the matrices $\{E_{ij} \mid 1 \leq i \leq l, 1 \leq j \leq k\}$ form a basis of the collection of all $l \times k$ matrices, the set $\{L_{ij}^{B,C} \mid 1 \leq i \leq l, 1 \leq j \leq k\}$ is a basis of $\mathrm{Hom}(V, W)$.