Contact: gwmelvin@colby.edu

March 15: Group Algebra

Convention: Unless otherwise specified, G will always denote a finite group, V a finite dimensional vector space over $\mathbb{C}$.

Proof of (8.5): (contd.)

- $\mathcal{S}$ is linearly independent: suppose we have a linear relation

$$\sum_{x \in G} a_x e_x = 0_{\mathbb{C}[G]}, \qquad \text{for some } a_x \in \mathbb{C},\, x \in G$$

  Then, for each $y \in G$,

$$\left(\sum_{x \in G} a_x e_x\right)(y) = 0 \quad \implies \quad \sum_{x \in G} a_x e_x(y) = 0 \quad \implies \quad a_y = 0$$

  Hence, $a_y = 0$, for every $y \in G$, and $\mathcal{S}$ is linearly independent.

- Consider

$$\langle , \rangle : \mathbb{C}[G] \times \mathbb{C}[G] \to \mathbb{C}\ ,\ (f, g) \mapsto \frac{1}{|G|} \sum_{x \in G} f(x)\overline{g(x)}$$

  Sesquilinearity of $\langle , \rangle$ is straightforward to show. For $f \in \mathbb{C}[G]$ we have

$$\langle f, f \rangle = \sum_{x \in G} |f(x)|^2 \geq 0$$

  Moreover,

$$\begin{aligned} \langle f, f \rangle = 0 \quad &\iff \quad |f(x)|^2 = 0, \quad \text{for every } x \in G \\ &\iff \quad f(x) = 0, \quad \text{for every } x \in G. \\ &\iff \quad f = 0_{\mathbb{C}[G]}, \quad \text{is the zero function.} \end{aligned}$$

QED

Remark (9.1): Let $\rho : G \to \mathrm{GL}_k(\mathbb{C})$ be a matrix representation of G. Denote the ij-entry of ρ_g by $\rho_{ij}(g)$. Then, $\rho_{ij} : G \to \mathbb{C}$ is a function i.e. $\rho_{ij} \in \mathbb{C}[G]$. We call ρ_{ij} a **matrix element of** ρ (or **matrix coefficient of** ρ).

Recall the **unitary group** of $k \times k$ matrices

$$U_k(\mathbb{C}) = \{A \in M_k(\mathbb{C}) \mid A^{-1} = \overline{A}^t\}$$

This is the group of all matrices such that $\langle Au, Av \rangle_s = \langle u, v \rangle_s$, for every $u, v \in \mathbb{C}^k$. Here $\langle , \rangle_s$ is the standard inner product on $\mathbb{C}^k$.

We aim to prove the following technical result:

Theorem: (Schur Orthogonality Relations)
Suppose that $\varphi : G \to U_k(\mathbb{C})$, $\rho : G \to U_l(\mathbb{C})$ are inequivalent irreducible unitary matrix representations of G. Then,

1. $\langle \varphi_{ij}, \rho_{sl} \rangle = 0$, for all i, j, r, s.

2. $\langle \varphi_{ij}, \varphi_{rs} \rangle = \begin{cases} \frac{1}{k}, & \text{if } i = r \text{ and } j = s, \\ 0, & \text{else} \end{cases}$

Before we proceed to prove the Schur Orthogonality Relations, let's see some consequences.

Corollary (9.2): *Let φ be an irreducible unitary representation of degree k. Then, the set*

$$\{\sqrt{k}\varphi_{ij} \mid 1 \leq i, j \leq k\} \subseteq \mathbb{C}[G]$$

is orthonormal.

Proof: This is an immediate consequence of 2. above: observe that

$$\langle \sqrt{k}\varphi_{ij}, \sqrt{k}\varphi_{rs} \rangle = k\langle \varphi_{ij}, \varphi_{rs} \rangle$$

QED

Corollary (9.3): *There are only finitely many equivalence classes of irreducible representations of G. In fact, let $\#\mathrm{Irrep}(G)$ denote the no. of distinct equivalence classes of irreducible representations. Then,*

$$\#\mathrm{Irrep}(G) \leq |G|$$

Proof: Suppose $\#\mathrm{Irrep}(G) > |G|$, possibly infinite. Suppose $\rho^{(1)}, \ldots \rho^{(|G|+1)}$ are inequivalent irreducible representations of degree $k_1, \ldots, k_{|G|+1}$. Since every representation is equivalent to a unitary representation, upon choosing orthonormal bases, we obtain inequivalent irreducible unitary matrix representations

$$\rho^{(t)} : G \to U_{k_t}(\mathbb{C})$$

Let

$$R_t = \{\rho_{ij}^{(t)} \mid 1 \leq i, j \leq k_t\}, \qquad t = 1, \ldots, |G| + 1$$

Then, by 2. of Schur Orthogonality Relations, we have each set R_t is orthogonal. Moreover, by 1., the set

$$R = R_1 \cup R_2 \cup \cdots \cup R_{|G|+1}$$

is orthogonal. Furthermore, $R_t \cap R_s = \varnothing$, for each $t \neq s$: otherwise, if $f \in R_t \cap R_s$, with $t \neq s$, then $\langle f, f \rangle = 0$, by 1. But, by 2., $\langle f, f \rangle = \frac{1}{k}$, which is absurd. Hence,

$$|R| = \sum_{t=1}^{|G|+1} |R_t| = \sum_{t=1}^{|G|+1} k_t^2 \geq \sum_{t=1}^{|G|+1} 1 = |G| + 1$$

i.e. we have found an orthogonal, hence linearly independent, set $R \subseteq \mathbb{C}[G]$ such that $|R| > |G| = \dim \mathbb{C}[G]$, which is impossible. Therefore, it can't be possible to find $|G| + 1$ inequivalent irreducible representations of G.

QED

Remark (9.4): In fact the proof tells us more: if $\rho_1, \ldots, \rho_{\#\mathrm{Irrep}(G)}$ are inequivalent reps, with ρ_i having degree k_i, then

$$\#\mathrm{Irrep}(G) \leq k_1^2 + \ldots + k_{\#\mathrm{Irrep}(G)}^2 \leq |G|$$