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March 13: Consequences of Schur's Lemma; Group Algebra

Convention: Unless otherwise specified, G will always denote a finite group, V a finite dimensional vector space over $\mathbb{C}$.

Corollary (8.1): *Let G be a finite abelian group. Then, any irreducible representation is 1-dimensional.*

Proof: Let (ρ, V) be irreducible. Then, for any $g \in G$, ρ_g is a G-morphism: indeed, for any $h \in G$,

$$\rho_g \rho_h = \rho_{gh} = \rho_{hg} = \rho_h \rho_g$$

Hence, for any $g \in G$, there exists a scalar $\lambda_g \in \mathbb{C}$ such that $\rho_g = \lambda \cdot \mathrm{id}_V$, by Schur's Lemma 7.2. In particular, let $v \in V$ be nonzero. Then, for all $g \in G$, $\rho_g(v) = \lambda_g v$ so that $\mathrm{span}(v) \subseteq V$ is a subrepresentation. Since V is irreducible, we get $\mathrm{span}(v) = V$.
QED

Corollary (8.2): *Let G be a finite abelian group, (ρ, V) a representation of V. Then, there exists a basis $B \subseteq V$ such that*

$$[\rho_g]_B = \begin{bmatrix} \lambda_1(g) & 0 & \cdots & 0 \\ 0 & \lambda_2(g) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_k(g) \end{bmatrix}, \qquad \text{for every } g \in G.$$

In particular, if $G \subseteq \mathrm{GL}_k(\mathbb{C})$ is a finite abelian group then there exists $P \in \mathrm{GL}_k(\mathbb{C})$ such that $P^{-1}gP = D_g$ is diagonal, for every $g \in G$ i.e. G is **simultaneously diagonalisable**.

Proof: By Maschke's Theorem (Corollary 6.3), we can find subrepresentations $U_1, \ldots, U_r$ such that $V = U_1 \oplus \cdots \oplus U_r$, with U_i irreducible. By Corollary 8.1, each U_i is 1-dimensional so that $U_i = \mathrm{span}(v_i)$ and $r = \dim V = k$. Let $B = (v_1, \ldots, v_k)$. Then, B is a basis of V and each v_i is a common eigenvector fo4 ρ_g, for every $g \in G$. Hence, for every $g \in G$, there are scalars $\lambda_1(g), \ldots, \lambda_k(g)$ such that $\rho_g(v_i) = \lambda_i(g)$. In particular,

$$[\rho_g]_B = \begin{bmatrix} \lambda_1(g) & 0 & \cdots & 0 \\ 0 & \lambda_2(g) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_k(g) \end{bmatrix}, \qquad \text{for every } g \in G.$$

Moreover, if $G \subseteq \mathrm{GL}_k(\mathbb{C})$ then consider $\rho : G \to \mathrm{GL}_k(\mathbb{C})$, $g \mapsto g$ to be the inclusion homomorphism. Then, by what we have just shown, there is a basis $B \subseteq \mathbb{C}^k$ such that, letting $P = P_{S \leftarrow B}$ (with $S \subseteq \mathbb{C}^k$ the standard basis), we have $P^{-1}gP = D_g$ is diagonal, for every $g \in G$.
QED

Corollary (8.3): (Application of Representation Theory to Linear Algebra)

Let $T \in \mathrm{End}(V)$ be a linear map satisfying $T^n = \mathrm{id}_V$, for some $n > 0$. Then, T is diagonalisable.

Proof: Define a map

$$\rho : \mathbb{Z}/n\mathbb{Z} \to \mathrm{GL}(V)\ ,\ \bar{j} \mapsto T^j$$

This map is well-defined and a homomorphism because $T^n = \mathrm{id}_V$. Hence, by Corollary 8.2, there is a basis $B \subseteq V$ such that $[T]_B = [\rho_{\bar{1}}]_B$ is diagonal. This is precisely what it means for T to be diagonalisable.

QED

Remark (8.4): Corollary 8.4 can be proved using the theory of the minimal polynomial of a linear map. Our approach above does not rely on the minimal polynomial, however.

Section IV: The Group Algebra

Definition (8.5): Define the **group algebra of** G to be

$$\mathbb{C}[G] = \{f : G \to \mathbb{C}\}$$

the set of *all functions* from G to $\mathbb{C}$. For $x \in G$, define $e_x \in \mathbb{C}[G]$ to be the function

$$e_x(y) = \begin{cases} 1, & \text{if } x = y, \\ 0, & \text{else.} \end{cases}$$

Proposition (8.7):

- *$\mathbb{C}[G]$ is a vector space over $\mathbb{C}$.*
- *$\mathcal{S} = \{e_x \mid x \in G\}$ is a basis of $\mathbb{C}[G]$.*
- *The function*

$$\mathbb{C}[G] \times \mathbb{C}[G] \to \mathbb{C}\ ,\ (f, g) \mapsto \frac{1}{|G|} \sum_{x \in G} f(x)\overline{g(x)}$$

defines an inner product $\langle , \rangle$ on $\mathbb{C}[G]$.

Proof:

- Define addition and scalar multiplication pointwise: for $f, g \in \mathbb{C}[G]$, $c \in \mathbb{C}$

$$(f + g)(x) = f(x) + g(x), \qquad (cf)(x) = cf(x), \qquad \text{for every } x \in G.$$

Also, $0_{\mathbb{C}[G]}$ is the zero function, $0_{\mathbb{C}[G]}(x) = 0$, $x \in G$. With these operations, the group algebra is a vector space over $\mathbb{C}$.

- $\mathcal{S}$ spans $\mathbb{C}[G]$: Let $f \in \mathbb{C}[G]$. Then, we claim that

$$f = \sum_{x \in G} f(x)e_x \in \mathbb{C}[G]$$

Indeed: let $y \in G$. Then,

$$\left(\sum_{x \in G} f(x)e_x\right)(y) = \sum_{x \in G} f(x)e_x(y) = f(y), \quad \text{by definition of } e_x.$$

Hence $\mathcal{S}$ spans.