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March 11: G-morphisms; Schur's Lemma

Convention: Unless otherwise specified, G will always denote a finite group, V a finite dimensional vector space over $\mathbb{C}$.

Proposition (7.1): *Let (ρ, V), (φ, W) be representations of G and $T : V \to W$ be a G-morphism. Then,*

1. $\ker T$ *is a subrepresentation of* V,
2. $\operatorname{im} T$ *is a subrepresentation of* W.

Proof:

1. Let $v \in \ker T$, $g \in G$. Then,
$$T(\rho_g(v)) = \varphi_g(T(v)) = \varphi_g(0_W) = 0_W \quad \implies \quad \rho_g(v) \in \ker T.$$
2. Let $w \in \operatorname{im} T$, $g \in G$. Suppose $w = T(v)$. Then,
$$\varphi_g(w) = \varphi_g(T(v)) = T(\rho_g(v)) \in \operatorname{im} T$$

QED

We have the following important consequence of Proposition 7.1:

Lemma (7.2): (Schur's Lemma)
Let $(\rho, V), (\varphi, W)$ be irreducible representations of G, $T \in \operatorname{Hom}_G(\rho, \varphi)$. Then, either $T = 0$ or T is invertible.

Moreover,

- *If φ, ρ are inequivalent then* $\operatorname{Hom}_G(\rho, \varphi) = 0$.
- *If $\varphi = \rho$ then* $\operatorname{End}_G(\rho) = \operatorname{Hom}_G(\rho, \rho) = \{\lambda \cdot \mathrm{id}_V \mid \lambda \in \mathbb{C}\}$.

Proof: Suppose that $T \neq 0$. Then,

- $\ker T \neq V$ is a subrepresentation of V, by Proposition 7.1. Hence, $\ker T = \{0_V\}$, since ρ irreducible, and T is injective.
- $\operatorname{im} T \neq \{0_W\}$ is a subrepresentation of W, by Proposition 7.1. Hence, $\operatorname{im} T = W$, since φ is irreducible, and T is surjective.

Therefore, if $T \neq 0$ then T is invertible.

Suppose now that $\rho = \varphi$. Let $T : V \to V$ be a G-morphism. Let $\lambda \in \mathbb{C}$ be an eigenvalue, $v \in \ker(T - \lambda \mathrm{id}_V)$ an associated eigenvector. Hence,
$$E_\lambda = \ker(T - \lambda \mathrm{id}_V) \neq \{0_V\}$$

Moreover, for any $g \in G$,

$$T(\rho_g(v)) = \rho(g)(T(v)) = \rho_g(\lambda v) = \lambda \rho_g(v) \quad \Longrightarrow \quad \rho_g(v) \in E_\lambda$$

Hence, E_λ is a nonzero subrepresentation and $E_\lambda = V$, since ρ is irreducible. In particular, for any $v \in V$, $T(v) = \lambda v$.

QED

Remark (7.3):

- $\mathrm{Hom}_G(\rho, \varphi) \subseteq \mathrm{Hom}(V, W)$ is a subspace.
- Schur's Lemma implies that $\mathrm{End}_G(\rho) = \mathrm{Hom}_G(\rho, \rho) \subseteq \mathrm{End}(V)$ is a subring. In fact, $\mathrm{End}_G(\rho)$ is, in a natural way, a field.