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Some thoughts and advice:

- Please submit solutions to the following problems by **Wednesday, March 20th, 1.10pm**. You can either submit your solution in class or leave it outside my office.
- You should expect to spend several hours on homework sets. A lot of practice problem-solving is essential to understand the material and skills covered in class. Be organised and do not leave problem sets until the last-minute. Instead, get a good start on the problems as soon as possible.
- When approaching a problem think about the following: *do you understand the words used to state the problem? what is the problem asking you to do? can you restate the problem in your own words? have you seen a similar problem worked out in class? is there a similar problem worked out in the textbook? what results/skills did you see in class that might be related to the problem?*

  If you are stuck for inspiration come to office hours, or send me an email. However, don't just ask for the solution - provide your thought process, the difficulties you are having, and ask a coherent question in complete English sentences.
- Form study groups - get together and work through problem sets. **This will make your life easier!** You must write your solutions *on your own* and *in your own words*.
- You **are not allowed** to use any additional resources (e.g. stackoverflow.com). If you are concerned then please ask.

G-morphisms

UNLESS OTHERWISE SPECIFIED, ALL VECTOR SPACES WILL HAVE SCALAR FIELD $\mathbb{C}$.

1. Let $T : V \to W$ be a G-morphism from (ρ, V) to (ϕ, W).
   (a) Prove that there exists a subrepresentation $U \subseteq V$ such that $V \simeq U \oplus \ker T$ and $U \simeq \operatorname{im} T$.
   (b) Deduce that if T is surjective then there exists a subrepresentation $U \subseteq V$ such that $U \simeq W$.

2. Let (ρ, V), (φ, W) and (ψ, U) be representations of G.
   (a) Show that
   $$\mathrm{Hom}_G(\rho, \varphi) = \{T : V \to W \mid \ T \text{ is a } G\text{-morphism}\}$$
   is a vector subspace of $\mathrm{Hom}(V, W) = \{T : V \to W \mid T \text{ linear}\}$.
   (b) Let $T \in \mathrm{Hom}_G(\rho, \varphi)$, $S \in \mathrm{Hom}_G(\varphi, \psi)$. Show that $S \circ T \in \mathrm{Hom}_G(\rho, \psi)$.
   (c) Deduce that $\mathrm{End}_G(\rho) = \mathrm{Hom}_G(\rho, \rho)$ is a subring of $\mathrm{End}(V)$.

   In the remainder of the Exercises we will describe the subring $\mathrm{End}_G(\rho)$ explicitly, for some representations ρ.

3. Consider the representation
   $$\rho : \mathbb{Z}/3\mathbb{Z} \to \mathrm{GL}_3(\mathbb{C}) \ , \ \bar{j} \mapsto \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^j$$
   In Homework 5, Exercise 6, you showed that $\mathbb{C}^3 = U_1 \oplus U_2 \oplus U_3$, with each $U_i = \mathrm{span}(v_i) \subseteq \mathbb{C}^3$ a degree 1 subrepresentation, and U_1, U_2, U_3 are mutually inequivalent. In particular, $B = (v_1, v_2, v_3) \subseteq V$ is a basis of common eigenvectors for all ρ_g, $g \in G$.

(a) Let $T \in \mathrm{End}_G(\rho)$ be a G-morphism. Suppose that

$$T(v_1) = a_1 v_1 + a_2 v_2 + a_3 v_3, \qquad a_1, a_2, a_3 \in \mathbb{C}$$

. Show that $a_2 = a_3 = 0$.

(b) Deduce that $[\rho_g]_B$ is diagonal, for every $g \in G$.

(c) Using the '*matrix with respect to* B' map $\mathrm{End}(V) \to M_3(\mathbb{C})$, $T \mapsto [T]_B$, show that there is an isomorphism of rings

$$\mathrm{End}_G(\rho) \simeq \mathbb{C} \times \mathbb{C} \times \mathbb{C}.$$

Here $\mathbb{C} \times \mathbb{C} \times \mathbb{C}$ is the product ring with component-wise addition and multiplication.

4. Let $\rho : \mathbb{Z}/n\mathbb{Z} \to \mathrm{GL}_n(\mathbb{C})$ be the representation from Homework 5, Exercise 6. Formulate and prove a conjecture identifying $\mathrm{End}_G(\rho) \simeq R$, where R is a more familiar (commutative) ring.