MA434: Spring 2019
Homework 5
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Some thoughts and advice:

- Please submit solutions to the following problems by **Wednesday, March 13th, 1.10pm**. You can either submit your solution in class or leave it outside my office.
- You should expect to spend several hours on homework sets. A lot of practice problem-solving is essential to understand the material and skills covered in class. Be organised and do not leave problem sets until the last-minute. Instead, get a good start on the problems as soon as possible.
- When approaching a problem think about the following: *do you understand the words used to state the problem? what is the problem asking you to do? can you restate the problem in your own words? have you seen a similar problem worked out in class? is there a similar problem worked out in the textbook? what results/skills did you see in class that might be related to the problem?*

  If you are stuck for inspiration come to office hours, or send me an email. However, don't just ask for the solution - provide your thought process, the difficulties you are having, and ask a coherent question in complete English sentences.
- Form study groups - get together and work through problem sets. **This will make your life easier!** You must write your solutions *on your own* and *in your own words*.
- You **are not allowed** to use any additional resources (e.g. `stackoverflow.com`). If you are concerned then please ask.

Reducibility, unitary representations.

UNLESS OTHERWISE SPECIFIED, ALL VECTOR SPACES WILL HAVE SCALAR FIELD $\mathbb{C}$.

1. Prove Lemmas 3.1.24, 3.1.25 in the textbook. See the proof of Lemma 3.1.23 for inspiration.
2. Let G be a finite group, $\mathcal{R}$ the collection of all finite dimensional complex representations of G. Show that $V \sim V'$ if and only if V is equivalent to V' defines an equivalence relation on $\mathcal{R}$.
3. Exercise 3.6 in the textbook, including the following additional problem: *Ex. 3.6.3: show that φ^χ is irreducible if and only if φ is irreducible.*
4. Exercise 3.8 in the textbook.
5. Let $\rho : G \to \mathrm{GL}(V)$ be a representation of the finite group G.
   (a) Let $U, W \subseteq V$ be subrepresentations. Show that $U \cap W$ and $U + W$ are subrepresentations.
   (b) Let $U, W \subseteq V$ be irreducible subrepresentations. Prove that either $U \cap W = \{0_V\}$ or $U = W$.
6. Let $X \in M_n(\mathbb{C})$ be the matrix
   $$X = [e_2\, e_3\, \cdots e_n\, e_1]$$
   Here $e_i \in \mathbb{C}^n$ is the i^{th} standard basis vector. Consider the map
   $$\rho : \mathbb{Z}/n\mathbb{Z} \to \mathrm{GL}_n(\mathbb{C})\ ,\ \overline{j} \mapsto X^j$$
   (a) Verify that ρ defines a homomorphism. (*Hint: utilise the standard representation $\varphi : S_n \to$* $\mathrm{GL}_n(\mathbb{C})$.)
   (b) Show that ρ decomposes as a direct sum of n inequivalent degree 1 representations. (*Hint: Exercise 3.4 will be useful here; also, it may help to experiment with the $n = 5$ case*)