MA434: Spring 2019
Homework 4

Contact: gwmelvin@colby.edu

Some thoughts and advice:

- Please submit solutions to the following problems by **Wednesday, March 6th, 1.10pm**. You can either submit your solution in class or leave it outside my office.
- You should expect to spend several hours on homework sets. A lot of practice problem-solving is essential to understand the material and skills covered in class. Be organised and do not leave problem sets until the last-minute. Instead, get a good start on the problems as soon as possible.
- When approaching a problem think about the following: *do you understand the words used to state the problem? what is the problem asking you to do? can you restate the problem in your own words? have you seen a similar problem worked out in class? is there a similar problem worked out in the textbook? what results/skills did you see in class that might be related to the problem?*

  If you are stuck for inspiration come to office hours, or send me an email. However, don't just ask for the solution - provide your thought process, the difficulties you are having, and ask a coherent question in complete English sentences.
- Form study groups - get together and work through problem sets. **This will make your life easier!** You must write your solutions *on your own* and *in your own words*.
- You **are not allowed** to use any additional resources (e.g. stackoverflow.com). If you are concerned then please ask.

Subrepresentations, irreducible representations

UNLESS OTHERWISE SPECIFIED, ALL VECTOR SPACES WILL HAVE SCALAR FIELD $\mathbb{C}$.

1. Exercise 3.3, 3.4 in the textbook. (*You've already done 3.4 (1) in Homework 3 so you can skip it.*)

2. Let V be a vector space over $\mathbb{C}$, and let $U, W \subseteq V$ be subspaces satisfying the following properties:

   - $V = U + W = \{u + w \mid u \in U, w \in W\}$,
   - $U \cap W = \{0_V\}$.

   Show that, for every $v \in V$, there exists unique $u \in U$, $w \in W$ such that $v = u + w$. We say that V is the **(internal) direct sum of** U, W.

3. Let V_1, V_2 be vector spaces over $\mathbb{C}$. The **(external) direct sum of** V_1, V_2 is the vector space $V_1 \oplus V_2$ defined as follows:

   - As a set $V_1 \oplus V_2 = V_1 \times V_2$, the Cartesian product.
   - Define addition component-wise: $(u_1, u_2) + (v_1, v_2) = (u_1 + v_1, u_2 + v_2)$, for any $(u_1, u_2), (v_1, v_2) \in V_1 \times V_2$.
   - Define scalar multiplication component-wise: $c(u_1, u_2) = (cu_1, cu_2)$, for any $c \in \mathbb{C}$, $(u_1, u_2) \in V_1 \times V_2$.
   - The zero vector is $(0_{V_1}, 0_{V_2})$.

   Let V be a vector space over $\mathbb{C}$, U, W subspaces of V satisfying the properties in the previous Exercise. Give a linear map

   $$T : U \oplus W \to V \ , \ (u, w) \mapsto T(u, w)$$

   that is an isomorphism of vector spaces. (*Hint: given two vectors in V how can you produce a third?*)

4. Let $G = S_3$, the symmetric group on 3 letters, and $\mathbb{C}^3$ be the standard representation.

   (a) Show that the subrepresentation $W = \operatorname{nul}([1\ 1\ 1]) = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \mid a+b+c=0 \right\}$ is irreducible.

   (b) Show that $\mathbb{C}^3 \simeq \mathbb{C} \oplus W$. Here $\mathbb{C}$ is the degree 1 trivial representation.

5. Let $\rho : G \to \mathrm{GL}_2(\mathbb{C})$ be a degree 2 representation of G. Suppose that $\rho \simeq \rho_1 \oplus \rho_2$, where ρ_1, ρ_2 are degree 1 representations of G. Show that $\rho_g \rho_h = \rho_h \rho_g$, for all $g, h \in G$.