Contact: gwmelvin@colby.edu

Some thoughts and advice:

- Please submit solutions to the following problems by **Wednesday, February 27th, 1.10pm**. You can either submit your solution in class or leave it outside my office.
- You should expect to spend several hours on homework sets. A lot of practice problem-solving is essential to understand the material and skills covered in class. Be organised and do not leave problem sets until the last-minute. Instead, get a good start on the problems as soon as possible.
- When approaching a problem think about the following: *do you understand the words used to state the problem? what is the problem asking you to do? can you restate the problem in your own words? have you seen a similar problem worked out in class? is there a similar problem worked out in the textbook? what results/skills did you see in class that might be related to the problem?*

  If you are stuck for inspiration come to office hours, or send me an email. However, don't just ask for the solution - provide your thought process, the difficulties you are having, and ask a coherent question in complete English sentences.
- Form study groups - get together and work through problem sets. **This will make your life easier!** You must write your solutions *on your own* and *in your own words*.
- You **are not allowed** to use any additional resources (e.g. stackoverflow.com). If you are concerned then please ask.

Inner product spaces; representations

Unless otherwise specified, all vector spaces will have scalar field $\mathbb{C}$.

1. Let $(V, \langle , \rangle)$ be an inner product space, $T \in \text{End}(V)$, and let $T^* \in \text{End}(V)$ be the adjoint of T. Show that $T^* \in \text{End}(V)$ is linear.

2. Let V be a finite dimensional vector space. Denote

$$V^* = \text{Hom}(V, \mathbb{C}) = \{L : V \to \mathbb{C} \mid \ L \text{ linear}\},$$

the *dual space of* V. V^* is a vector space over $\mathbb{C}$: for any $K, L \in V^*$, $c \in \mathbb{C}$, define $K + L \in V^*$ and $cL \in V^*$ as follows

$$(K + L)(v) = K(v) + L(v), \quad v \in V,$$
$$(cL)(v) = cL(v), \quad v \in V.$$

In class, when $(V, \langle , \rangle)$ is an inner product space, we used the inner product to determine a bijection $\alpha : V \to V^*$.

(a) Show that $\alpha(cv) \neq c\alpha(v)$, for $c \in \mathbb{C}$, $v \in V$. Deduce that α is not linear. Hence, α is not an *isomorphism of vector spaces.*

(b) Given $v \in V$, define the *evaluation at* v function

$$E_v : V^* \to \mathbb{C}\ ,\ L \mapsto L(v)$$

Remember: $L : V \to \mathbb{C}$ is a linear function.

i. Show that E_v is linear. Hence, $E_v \in \text{Hom}(V^*, \mathbb{C}) = (V^*)^*$.

ii. Consider the function

$$E : V \to (V^*)^* \ , \ v \mapsto E_v$$

Show that E is linear and that E is injective.
We say that E is a *natural embedding of V in its double dual* $(V^*)^*$: there is no arbitrary choice (of a basis, say) required to construct this embedding.

3. Let $G = \mathbb{Z}/3\mathbb{Z}$ and define the degree three representation ρ on $\mathbb{C}^3$ by

$$\rho(\overline{0}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} , \quad \rho(\overline{1}) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} , \quad \rho(\overline{2}) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

   (a) Find a one-dimensional subrepresentation $U \subseteq \mathbb{C}^3$.
   (b) Can you find another one-dimensional subrepresentation? (*Hint: what is the relationship between one-dimensional subrepresentations and eigenvectors?*)

4. In this problem you will investigate a finite dimensional representation of an infinite group.
   (a) Show that

$$\rho : \mathbb{Z} \to \mathrm{GL}_2(\mathbb{C}) \ , \ n \mapsto \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$$

   defines a representation of $(\mathbb{Z}, +)$.
   (b) Show that $U = \mathrm{span}(e_1) \subseteq \mathbb{C}^2$ is a subrepresentation equivalent to the trivial representation. (*You will need to construct a linear isomorphism $f : U \to \mathbb{C}$ with the appropriate properties*)
   (c) Is it possible to find a subrepresentation $W \subseteq \mathbb{C}^2$ such that
      - $W \cap U = \{\underline{0}\}$, and
      - $W + U = \mathbb{C}^2$?

   If so, give an example of such a subrepresentation W, making sure to verify that W satisfies the required properties: if not, explain.

5. Here is a FUN FACT: *Let $T \in \mathrm{End}(V)$ be a linear map, V finite dimensional over $\mathbb{C}$. Then, the roots of m_T, the minimal polynomial of T, are precisely the eigenvalues of T.*

   Let $\rho : G \to \mathrm{GL}(V)$ be a representation of finite degree of the finite group G. Using the FUN FACT, show that the eigenvalues of $\rho(g)$, $g \in G$, are roots of unity.