Contact: `gwmelvin@colby.edu`

Some thoughts and advice:

- Please submit solutions to the following problems by **Wednesday, February 20th, 1.10pm**. You can either submit your solution in class or leave it outside my office.
- You should expect to spend several hours on homework sets. A lot of practice problem-solving is essential to understand the material and skills covered in class. Be organised and do not leave problem sets until the last-minute. Instead, get a good start on the problems as soon as possible.
- When approaching a problem think about the following: *do you understand the words used to state the problem? what is the problem asking you to do? can you restate the problem in your own words? have you seen a similar problem worked out in class? is there a similar problem worked out in the textbook? what results/skills did you see in class that might be related to the problem?*

  If you are stuck for inspiration come to office hours, or send me an email. However, don't just ask for the solution - provide your thought process, the difficulties you are having, and ask a coherent question in complete English sentences.
- Form study groups - get together and work through problem sets. **This will make your life easier!** You must write your solutions *on your own* and *in your own words*.
- You **are not allowed** to use any additional resources (e.g. `stackoverflow.com`). If you are concerned then please ask.

Further linear algebra review

Unless otherwise specified, all vector spaces will have scalar field $\mathbb{C}$.

1. Problem 2.1 in the textbook.
2. Problem 2.5 in the textbook.
3. Consider the matrix

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

   Determine an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

4. Let V be a k-dimensional vector space. Define the *endomorphism ring of V* to be

$$\mathrm{End}(V) = \{T : V \to V \mid T \text{ linear}\}$$

   - $\mathrm{End}(V)$ is a vector space over $\mathbb{C}$: for $T, S \in \mathrm{End}(V)$, we define $T+S \in \mathrm{End}(V)$ to be the linear map $(T+S)(v) = T(v) + S(v)$; for any scalar $c \in \mathbb{C}$ and linear map $T \in \mathrm{End}(V)$, define $cT \in \mathrm{End}(V)$ to be the linear map $(cT)(v) = cT(v)$.
   - $\mathrm{End}(V)$ is a ring: for $T, S \in \mathrm{End}(V)$ we define a multiplication $*$ on $\mathrm{End}(V)$ as follows: $T*S = T \circ S$ is composition of functions.

   (a) Prove that $\mathrm{End}(V)$ is isomorphic (as a vector space) to $M_k(\mathbb{C})$, the vector space of $k \times k$ matrices with entries in $\mathbb{C}$. (*Hint: choose a basis $B \subseteq V$*)

(b) (Problem 2.2 in textbook) Let $B \subseteq V$ be a basis. Prove that the function

$$\text{End}(V) \to M_k(\mathbb{C}) \ , \ T \mapsto [T]_B$$

is an isomorphism of rings.

5. Denote the polynomial algebra in one variable with complex coefficients by $\mathbb{C}[t]$. For example,

$$\sqrt{-1}t^5 - \pi t^2 + 2 \in \mathbb{C}[t].$$

In your previous linear algebra/abstract algebra courses you will have obtained the following properties of $\mathbb{C}[t]$:

- $\mathbb{C}[t]$ is a vector space over $\mathbb{C}$;
- $\mathbb{C}[t]$ is a ring;
- $\mathbb{C}[t]$ is a principal ideal domain: if $I \subseteq \mathbb{C}[t]$ is an ideal then there exists $f \in I$ so that

$$I = (f) \stackrel{def}{=} \{fp \mid p \in \mathbb{C}[t]\}$$

In words: elements of I are multiples of f.
Note: since $\mathbb{C}[t]$ is a commutative ring, left ideals are right ideals and vice versa.

(a) Let $I \subseteq \mathbb{C}[t]$ be an nonzero ideal. Show that there exists a *unique* polynomial $g \in \mathbb{C}[t]$ having leading coefficient 1 so that $I = (g)$. We call g the **monic generator of** I.

(b) Let $T \in \text{End}(V)$ be a linear map, $T \neq \text{id}_V$. Show that the function

$$\rho_T : \mathbb{C}[t] \to \text{End}(V) \ , \ p(t) \mapsto p(T)$$

is

i. a linear map;
ii. a ring homomorphism.

(c) Prove that $\ker \rho_T \neq \{0\} \subseteq \mathbb{C}[t]$. (*Hint: injective linear maps take linearly independent sets to linearly independent sets.*)
Define $m_T \in \mathbb{C}[t]$ to be the monic generator of $\ker \rho_T$: we call m_T the **minimal polynomial of** T.

(d) Suppose that $T \in \text{End}(V)$ is diagonalisable. Prove that m_T is a product of a distinct linear factors. (*Hint: choose a basis of V consisting of eigenvectors of T*)

(e) In fact, **the converse is also true:** if m_T is a product of distinct linear factors then T is diagonalisable (you do not need to show this).
Using this fact, prove that if $T \in \text{End}(V)$ satisfies $T^m = \text{id}_V$, for some integer $m > 0$, then T is diagonalisable.

6. Updated 2/17: Problem 2.8 in the textbook using the following hint: *Let $v \in \mathbb{C}^n$ be an eigenvector of A and extend (v) to a basis $B \subseteq \mathbb{C}^n$.*

Deduce the following: let $T \in \text{End}(V)$. Then, there exists a basis $B \subseteq V$ such that $[T]_B$ is upper-triangular.