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Some thoughts and advice:

- Please submit solutions to the following problems by **Wednesday, February 13th, 1.10pm**. You can either submit your solution in class or leave it outside my office.
- You should expect to spend several hours on homework sets. A lot of practice problem-solving is essential to understand the material and skills covered in class. Be organised and do not leave problem sets until the last-minute. Instead, get a good start on the problems as soon as possible.
- When approaching a problem think about the following: *do you understand the words used to state the problem? what is the problem asking you to do? can you restate the problem in your own words? have you seen a similar problem worked out in class? is there a similar problem worked out in the textbook? what results/skills did you see in class that might be related to the problem?*

  If you are stuck for inspiration come to office hours, or send me an email. However, don't just ask for the solution - provide your thought process, the difficulties you are having, and ask a coherent question in complete English sentences.
- Form study groups - get together and work through problem sets. **This will make your life easier!** You must write your solutions *on your own* and *in your own words*.
- You **are not allowed** to use any additional resources (e.g. `stackoverflow.com`). If you are concerned then please ask.

Some linear algebra review

Unless otherwise specified, all vector spaces will have scalar field $\mathbb{C}$.

1. (a) Verify that the set
$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\} \subseteq \mathbb{C}^3$$
is a basis.

   (b) Determine the S-coordinates of $v = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in \mathbb{C}^3$.

   (c) Determine the S-coordinates of an arbitrary vector $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in \mathbb{C}^3$. (*Your solution will be in terms of* a_1, a_2, a_3)

   (d) Determine an explicit formula for the S-coordinate map $[-]_S : \mathbb{C}^3 \to \mathbb{C}^3$.

2. Let $T : V \to W$ be a linear map.

   (a) Define the *kernel of* T to be the subset
$$\ker T = \{v \in V \mid T(v) = 0_W \in W\} \subseteq V$$
Prove that $\ker T$ is a subspace of V.

(b) Define the *image of* T to be the subset

$$\text{im}\, T = \{w \in W \mid w = T(v) \text{ for some } v \in V\} \subseteq W$$

Prove that $\text{im}\, T$ is a subspace of W.

The following result is one of the most important theorems of linear algebra (it can be considered the Fundamental Theorem of Linear Maps).

Rank Theorem: Let $T : V \to W$ be a linear map, V finite dimensional. Then,

$$\dim \ker T + \dim \text{im}\, T = \dim V$$

3. Consider the subset

$$U = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \middle| \; a_1 + a_2 + a_3 + a_4 = 0 \right\} \subseteq \mathbb{C}^4$$

(a) Determine a linear map $T : \mathbb{C}^4 \to \mathbb{C}$ so that $U = \ker T$. Conclude that $U \subseteq \mathbb{C}^4$ is a subspace.

(b) Show that $\text{im}\, T = \mathbb{C}$. Deduce that $\dim U = 3$.

(c) Find a basis $\mathcal{B}$ of U. Make sure you verify that the set you give is, in fact, a basis. (*There are an infinite number of possibilities here!*)

4. Consider the linear map

$$L : \mathbb{C}^4 \to \mathbb{C}^4 \; , \quad \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \mapsto \begin{bmatrix} a_2 \\ a_3 \\ a_4 \\ a_1 \end{bmatrix}$$

Recall the subspace $U \subseteq \mathbb{C}^4$ and your basis $\mathcal{B}$ from the previous problem.

(a) Show that $L(u) \in U$, for any $u \in U$. Deduce that L *restricts* to a linear map

$$L_{|U} : U \to U \; , \; u \mapsto L(u)$$

(*The last part is quite straightforward (i.e. one or two sentences of explanation) as there's only two things to consider: (1)* $L_{|U}$ *is well-defined, (2)* $L_{|U}$ *is linear*)

(b) Determine the matrix of $L_{|U}$ with respect to $\mathcal{B}$; your resulting matrix will be 3×3.