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FEBRUARY 8: LINEAR ALGEBRA REVIEW

Let V be a vector space with field of scalars $\mathbb{C}$, $S = \{v_1, \ldots, v_k\} \subset V$ a subset of vectors. We recall the following fundamental notions:

- $\text{span}(S) = \{a_1v_1 + \ldots + a_nv_n \mid a_1, \ldots, a_n \in \mathbb{C}\}$, the set of all linear combinations of elements in S, is a subspace; $\dim S \leq n$.
- If $\text{span}(S) = V$ then we say S **spans** V: this means that, for every $v \in V$, there's $a_1, \ldots, a_n \in \mathbb{C}$ so that

  $$v = a_1v_1 + \ldots + a_nv_n$$

  For example, if $V = \mathbb{C}^k$ and $M = [v_1 \cdots v_n]$ is a $k \times n$ matrix having i^{th} column v_i then S spans $V \Leftrightarrow M$ has RREF containing a leading 1 in every row.
- Say S is **linearly dependent** if there's scalars $a_1, \ldots, a_n \in \mathbb{C}$, at least one of which is nonzero, such that

  $$a_1v_! + \ldots + a_nv_n = 0_V \in V$$

  For example, if $V = \mathbb{C}^k$, M as above, then S linearly dependent $\Leftrightarrow$ there exists $x \in \mathbb{C}^n$, $x \neq 0$, so that $Mx = 0 \in \mathbb{C}^k$.
- S is **linearly independent** if it's **not** linearly dependent. This means that whenever we have a linear relation

  $$a_1v_1 + \ldots + a_nv_n = 0_V$$

  it must be the case that $a_1 = \ldots = a_n = 0 \in \mathbb{C}$. For example, if $V = \mathbb{C}^n$, M as above, then S linearly independent $\Leftrightarrow$ M has RREF containing a leading 1 in every column.
- Say S is a **basis** (of V) if $\text{span}(S) = V$ and S is linearly independent. This means that, for every $v \in V$, there exists *unique* scalars $a_1, \ldots, a_n \in \mathbb{C}$ such that

  $$v = a_1v_1 + \ldots + a_nv_n$$
- Define the **dimension of** V, denoted $\dim V$, to be the cardinality of any basis (it's a basic fact that this definition is well-defined).

COORDINATES

Let $B = (v_1, \ldots, v_k) \subset V$ be a basis of V. For any $v \in V$, define the B**-coordinates** of v to be

$$[v]_B = \begin{bmatrix} a_1 \\ \vdots \\ a_k \end{bmatrix} \in \mathbb{C}^k$$

where $a_1, \ldots, a_k$ are the unique scalars such that $v = \sum_{i=1}^{k} a_i v_i$.

Example: Let

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Then, B is a basis of $\mathbb{C}^3$. Let $v = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$: what's $[v]_B$? We must find a_1, a_2, a_3 satisfying

$$a_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + a_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

This is the same as solving the matrix equation

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Using row-reduction, we find

$$[v]_B = \begin{bmatrix} 1/2 \\ 0 \\ -1/2 \end{bmatrix};$$

that is,

$$(1/2) \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + (-1/2) \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Let $T : V \to W$ be a linear map, $\dim V = k, \dim W = l$, $B = (v_1, \ldots, v_k) \subset V$ and $C \subset W$ bases. Define the **matrix of T with respect to** B, C to be the $l \times k$ matrix

$$[T]_B^C = [[T(v_1)]_C \cdots [T(v_k)]_B]$$

When $V = W$ and $B = C$, write $[T]_B = [T]_B^B$. $[T]_B^C$ is the *unique* $l \times k$ such that, for any $v \in V$,

$$[T(v)]_C = [T]_B^C [v]_B, \qquad \text{(AMAZING PROPERTY)}$$