Representation Theory
Fall 2019
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February 27: Irreducibility, direct sum, decomposability

Convention: Unless otherwise specified, G will always denote a finite group, V a finite dimensional vector space over $\mathbb{C}$.

Remark (3.1):

- Let (ρ, V) be a representation of G, $U \subseteq V$ a subrepresentation. Choose a basis $B' \subseteq U$ and extend to a basis B of V. Then, for any $g \in G$,

$$[\rho_g]_B = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$$

is *block upper-triangular*, where the top left $*$ denotes a $\dim U \times \dim U$ matrix.

Conversely, if $B = (v_1, \ldots, v_k)$ is a basis of V such that

$$[\rho_g]_B = \begin{bmatrix} \overset{i \times i}{*} & * \\ 0 & * \end{bmatrix}, \qquad \text{for all } g \in G,$$

then $U = \text{span}(v_1, \ldots, v_i)$ is a subrepresentation of V.

- Suppose (ρ_1, V_1) and (ρ_2, V_2) are equivalent representations. Then:
  - $\dim V_1 = \dim V_2$
  - If $T : V_1 \to V_2$ is a G-isomorphism - $T \circ \rho_1(g) = \rho_2(g) \circ T$, for all $g \in G$ - and $B_1 \subseteq V_1$, $B_2 \subseteq V_2$ are bases, then

$$[T]_{B_1}^{B_2}[\rho_1(g)]_{B_1} = [\rho_2(g)]_{B_2}[T]_{B_1}^{B_2}$$

Conversely, if $\rho_1 : G \to \text{GL}_k(\mathbb{C})$ and $\rho_2 : G \to \text{GL}_k(\mathbb{C})$ and $P \in \text{GL}_k(\mathbb{C})$ satisfies

$$P^{-1}\rho_1(g)P = \rho_2(g), \qquad \text{for all } g \in G,$$

then ρ_1 and ρ_2 are equivalent: the linear map

$$T : \mathbb{C}^k \to \mathbb{C}^k \ , \ v \mapsto Pv$$

is a G-isomorphism.

In particular, $\rho_1 \simeq \rho_2$ if and only if there exists bases $B_1 \subseteq V_1$, $B_2 \subseteq V_2$, such that

$$[\rho_1(g)]_{B_1} = [\rho_2(g)]_{B_2}, \qquad \text{for all } g \in G.$$

Definition (3.2): Let (ρ, V) be a nonzero representation of G. We say that (ρ, V) is **irreducible** (or **simple**) if the only subrepresentations of V are $\{0_V\}$ or V.

Example (3.3):

1. Any degree 1 representation is irreducible.
2. The standard representation $\varphi : S_n \to \mathrm{GL}_n(\mathbb{C})$ is not irreducible: $U = \mathrm{span}(e_1 + \ldots + e_n) \subseteq \mathbb{C}^n$ is a subrepresentation.
3. The representation of D_8 defined by

$$\begin{aligned} \rho : \quad D_8 &\to GL_2(\mathbb{C}) \\ r &\mapsto \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\ s &\mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \end{aligned}$$

   is irreducible: if not, then there would exist a degree 1 subrepresentation $U = \mathrm{span}(v) \subseteq \mathbb{C}^2$. In particular, there would exist scalars $\lambda, \mu \in \mathbb{C}$ such that

$$rv = \lambda v, \qquad sv = \mu v \qquad (*)$$

   Now, s admits two distinct eigenvalues $1, -1$ with corresponding eigenspaces

$$E_1 = \mathrm{span}(e_1), \qquad E_{-1} = \mathrm{span}(e_2)$$

   This means that $v = ae_1$ or $v = be_2$. However, neither e_1 nor e_2 are eigenvectors for r so that $(*)$ can't hold. Hence, no such degree 1 subrepresentation U can exist so that ρ is irreducible.

Remark (3.4): Checking whether a given representation is irreducible is difficult, in general.

Definition (3.5): Let (ρ_1, V_1), (ρ_2, V_2) be representations of G. The **direct sum representation** is the representation

$$\rho_1 \oplus \rho_2 : G \to \mathrm{GL}(V_1 \times V_2)\ ,\ g \mapsto (\rho_1(g), \rho_2(g))$$

where, for $g \in G$, $(u, v) \in V_1 \times V_2$,

$$(\rho_1(g), \rho_2(g))(u, v) = (\rho_1(g)(u), \rho_2(g)(v))$$

This notion can be extended to define the direct sum $\rho_1 \oplus \cdots \oplus \rho_r$ of several representations $(\rho_1, V_1), \ldots, (\rho_r, V_r)$.

We say that (ρ, V) is **completely reducible** if ρ is equivalent to a direct sum of irreducible representations.

Remark (3.6): The product $V_1 \times V_2$ is a vector space as follows:

- $(v_1, v_2) + (u_1, u_2) = (v_1 + u_1, v_2 + u_2)$,
- $c \cdot (v_1, v_2) = (cv_1, cv_2)$,
- $0_{V_1 \times V_2} = (0_{V_1}, 0_{V_2})$.