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February 25: Subrepresentations, equivalence

Convention: Unless otherwise specified, G will always denote a finite group, V a finite dimensional vector space over $\mathbb{C}$.

Example (2.1): Let $G = S_n$. The **standard representation** is

$$\varphi : S_n \to \mathrm{GL}_n(\mathbb{C}) \ , \ \sigma \mapsto [e_{\sigma(1)} \ e_{\sigma(2)} \ \cdots \ e_{\sigma(n)}]$$

Here $e_i \in \mathbb{C}^n$ is the i^{th} standard basis vector of $\mathbb{C}^n$. The matrix φ_σ is the permutation matrix corresponding to σ.

Observe: for $\sigma, \tau \in S_n$, *the i^{th} column of φ_σ is $e_{\sigma(i)}$*. Hence, the i^{th} column of $\varphi_\tau\varphi_\sigma$ is the $\sigma(i)^{th}$ column of φ_τ i.e.

$$\varphi_\tau\varphi_\sigma = [e_{\tau(\sigma(1))} \ e_{\tau(\sigma(2))} \ \cdots \ e_{\tau(\sigma(n))}] = \varphi_{\tau\sigma}$$

Hence, φ is a homomorphism.

Definition (2.2): Let (ρ, V) be a representation of G, $U \subseteq V$ a subspace. We say that U is a **subrepresentation of** ρ (or, a **subrepresentation of** V) if $\rho_g(u) \in U$, for every $u \in U$, $g \in G$. We will also say that U is *G-invariant*.

Remark (2.3): If $U \subseteq V$ is a subrepresentation then ρ restricts to give a representation of G in U:

$$\rho_{|_U} : G \to \ , \ g \mapsto \rho(g)_{|_U}$$

Example (2.4): Let $U = \mathrm{span}(e_1 + \ldots + e_n) \subseteq \mathbb{C}^n$. Then, for any $\sigma \in S_n$,

$$\varphi_\sigma(e_1 + \ldots + e_n) = e_{\sigma(1)} + \ldots + e_{\sigma(n)} \in U$$

Hence, U is a subrepresentation of the standard representation of S_n.

There is another subrepresentation

$$W = \mathrm{nul}\left([1 \ 1 \ \cdots \ 1]\right) = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \mid x_1 + \ldots + x_n = 0 \right\}$$

Indeed, for $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in W$, $\sigma \in S_n$,

$$\varphi_\sigma(x) = x_1 e_{\sigma(1)} + x_2 e_{\sigma(2)} + \ldots + x_n e_{\sigma(n)} = \begin{bmatrix} x_{\sigma^{-1}(1)} \\ \vdots \\ x_{\sigma^{-1}(n)} \end{bmatrix}$$

Since the entries of $\varphi_\sigma(x)$ are just a permutation of the entries of x, their sum is also $0 \implies \varphi_\sigma(x) \in W$.

Remark (2.5): With respect to the basis $B = \left(\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}\right) \subseteq U$, we have $[\varphi_\sigma]_B = [1]$, for every $\sigma \in S_n$. Observe the similarity with the trivial representation.

Definition (2.6): Let (ρ_1, V_1) and (ρ_2, V_2) be representations of G. We say that ρ_1, ρ_2 are **equivalent** if there exists a linear isomorphism

$$T : V_1 \to V_2$$

such that $T \circ \rho_1(g) = \rho_2(g) \circ T$, for every $g \in G$. In this case, we write $\rho_1 \simeq \rho_2$ and say that T **intertwines** ρ_1, ρ_2, or that T is a G**-isomorphism**.

Example (2.7):

1. Consider the map

$$T : U \to \mathbb{C}\ , \quad \begin{bmatrix} a \\ \vdots \\ a \end{bmatrix} \mapsto a$$

   Then, T is a linear isomorphism. Also, for any $\sigma \in S_n$,

$$T\left(\varphi_\sigma\left(\begin{bmatrix} a \\ \vdots \\ a \end{bmatrix}\right)\right) = T\left(\begin{bmatrix} a \\ \vdots \\ a \end{bmatrix}\right) = a = \text{triv}_\sigma\left(T\left(\begin{bmatrix} a \\ \vdots \\ a \end{bmatrix}\right)\right)$$

   so that $T \circ \varphi_\sigma = \text{triv} \circ T$. Hence, the subrepresentation U is equivalent to the degree 1 trivial representation of S_n.

2. Let $G = D_8$, the symmetries of the unit square in $\mathbb{R}^2$. Let $r =$ rotation of the plane by $\pi/2$, $s =$ reflection across the x-axis. We have

$$D_8 = \{e, r, r^2, r^3, s, sr, sr^2, sr^3\}$$

   Then, realising these isometries as 2×2 matrices defines a representation:

$$\begin{aligned} \rho_1 :\ D_8 &\to GL_2(\mathbb{C}) \\ r &\mapsto \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\ s &\mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \end{aligned}$$

   Note that, in D_8, we have the relations

$$r^4 = s^2 = e, \quad sr = r^{-1}s$$

   You can check that

$$\rho_1(r)^4 = \rho_1(s)^2 = \mathbb{I}_2, \quad \text{and} \quad \rho_1(s)\rho_1(r) = \rho_1(r)^{-1}\rho_1(s)$$

   This ensures that ρ_1 defines a homomorphism.

We define another representation

$$\begin{aligned} \rho_2 : \quad D_8 &\rightarrow GL_2(\mathbb{C}) \\ r &\mapsto \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \\ s &\mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{aligned}$$

Claim: $\rho_1 \simeq \rho_2$.

Proof: Define $T : \mathbb{C}^2 \rightarrow \mathbb{C}^2$, $v \mapsto \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} v$. Then,

$$T(\rho_1(r) \left(\begin{bmatrix} a \\ b \end{bmatrix} \right)) = \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

and

$$\rho_2(r)(T \left(\begin{bmatrix} a \\ b \end{bmatrix} \right)) = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

Can check that

$$\begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}$$

so that T is a G-isomorphism..