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February 22: Representation Theory

Note: The completion of the proof of the Spectral Theorem is in February 20 lecture.

Convention: Unless otherwise specified, G will always denote a finite group, V a finite dimensional vector space over $\mathbb{C}$.

Definition (1.1): A **(linear) representation of** G is a group homomorphism

$$\rho : G \to \mathrm{GL}(V) = \{L \in \mathrm{End}(V) \mid \ L \text{ invertible}$$

The **degree of** ρ is $\dim V$. This means: for every $g, g' \in G$, $v, v' \in V$, $\lambda \in \mathbb{C}$,

- $\rho(g)(v + v') = \rho(g)(v) + \rho(g)(v')$, and $\rho(g)(\lambda v) = \lambda\rho(g)(v)$,
- $\rho(gg') = \rho(g) \circ \rho(g')$, $\rho(e_G) = \mathrm{id}_V$, and $\rho(g^{-1}) = \rho(g)^{-1}$.

Notation & Terminology (1.2): We usually write $\rho_g = \rho(g)$, $g \in G$, i.e. ρ_g is the linear map corresponding to $g \in G$.

We will often write (ρ, V) when we want to be explicit. By abuse of notation, we will also call V a **representation of** G i.e. we will implicitly assume the existence of the homomorphism ρ.

Remark (1.3): Identifying $\mathrm{GL}(V) \simeq \mathrm{GL}_k(\mathbb{C})$ (by choosing a basis B of V, say), we can identify the linear maps ρ_g, $g \in G$, with invertible $k \times k$ matrices.

When $V = \mathbb{C}^k$ we will frequently define ρ_g via its *standard matrix* (the matrix $[\rho_G]_S$) and define (by abuse of notation) a representation as a homomorphism

$$\rho : G \to \mathrm{GL}_k(\mathbb{C}).$$

Example:

1. Let $G = (\mathbb{Z}/3\mathbb{Z}, +)$. Define the degree 2 representation

$$\rho : G \to \mathrm{GL}_2(\mathbb{C}) \ , \ \overline{j} \mapsto \begin{bmatrix} \cos(2\pi j/3) & -\sin(2\pi j/3) \\ \sin(2\pi j/3) & \cos(2\pi j/3) \end{bmatrix}$$

   Here $\overline{j} = j + 3\mathbb{Z}$.

   This map is well-defined: if $\overline{j} = \overline{j'}$ then $j = j' + 3r$, for some integer r. Then

$$\begin{bmatrix} \cos(2\pi j/3) & -\sin(2\pi j/3) \\ \sin(2\pi j/3) & \cos(2\pi j/3) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(2\pi(j'+3)/3) & -\sin(2\pi(j'+3)/3) \\ \sin(2\pi(j'+3)/3) & \cos(2\pi(j'+3)/3) \end{bmatrix} = \begin{bmatrix} \cos(2\pi j'/3) & -\sin(2\pi j'/3) \\ \sin(2\pi j'/3) & \cos(2\pi j'/3) \end{bmatrix}$$

   Using double angle trig. formulae, you can show that $\rho(\overline{i} + \overline{j}) = \rho(\overline{i})\rho(\overline{j})$.

2. For any group G, define the degree 1 **trivial representation**

$$\mathrm{triv}_G : G \to \mathrm{GL}(\mathbb{C}) \ , \ g \mapsto \mathrm{id}_{\mathbb{C}}$$

i.e. for any $a \in \mathbb{C}$, $g \in G$, $\mathrm{triv}_G(g)(a) = a$.

3. Let S_n be the symmetric group on n letters. We will realise S_n as the group of bijections on $\{1, \dots, n\}$ so that elements of S_n are *functions*.

   Define the **standard representation of** S_n,

$$\varphi : S_n \to \mathrm{GL}_n(\mathbb{C}) \ , \ \sigma \mapsto \varphi_\sigma = [e_{\sigma(1)} \ e_{\sigma(2)} \ \cdots \ e_{\sigma(n)}]$$

   For example, when $n = 3$, we have

$$\varphi((123)) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \qquad \varphi((13)) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \qquad \varphi((12)) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{etc.}$$

   Let's check that φ is a homomorphism: we need to show that, for any $\sigma, \tau \in S_n$, $\varphi_{\tau\sigma} = \varphi_\tau \varphi_\sigma$.

   The i^{th} column of $\varphi_\tau \varphi_\sigma$ is obtained by multiplying the i^{th} column of φ_σ by φ_τ: that is, $\varphi_\tau e_{\sigma(i)}$. Thinking a little about how matrix multiplication works, we see that $\varphi_\tau e_{\sigma(i)}$ is the $\sigma(i)^{th}$ column of φ_τ, namely $e_{\tau(\sigma(i))}$. But this is precisely the i^{th} column of $\varphi_{\tau\sigma}$. Hence, $\varphi_{\tau\sigma} = \varphi_\tau \varphi_\sigma$.