Contact: gwmelvin@colby.edu

February 20: The Spectral Theorem

Today we prove the following:

Theorem: (Spectral Theorem) *Let* $(V, \langle,\rangle)$ *be an inner product space,* $T \in \mathrm{End}(V)$ *self-adjoint. Then, there exists an orthonormal basis of* V *consisting of eigenvectors of* T.

Remark: Additional information on the **orthogonal complement** $W^\perp$ of a subspace W can be found on the website.

Proof: We proceed by induction on $k = \dim V$.

$k = 1$: In this case, any $v \neq 0_V$, satisfying $||v|| = 1$, defines an orthonormal basis of V, namely $B = (v)$. Moreover, v is onviously an eigenvector i.e. $[T]_B = [\lambda]_B$, for some $\lambda \in \mathbb{C}$.

Suppose the statement of the theorem holds for all $1 \leq k < r$: if $\dim V = k$, $T \in \mathrm{End}(V)$ is self-adjoint, then there exists an orthonormal basis of V consisting of eigenvectors of T.

Suppose $k = \dim V = r > 1$. Let $\lambda \in \mathbb{C}$ be an eigenvalue of T (which must exist).

- (Case 1) If $E_\lambda = \ker(T - \lambda\mathrm{id}_V) = V$ then all nonzero vectors in V are eigenvectors of T, so we can choose *any* orthonormal basis $B \subset V$.

- (Case 2) Suppose that $E_\lambda \neq V$. Then, we have

$$V = E_\lambda + E_\lambda^\perp \qquad \text{and} \qquad E_\lambda \cap E_\lambda^\perp = \{0_V\}$$

  In particular, $1 \leq \dim E_\lambda^\perp < r$.

  Lemma 1: *For all* $v \in E_\lambda^\perp$, $T(v) \in E_\lambda^\perp$.

  Proof of Lemma 1: Let $v \in E_\lambda^\perp$. Then, for any $w \in E_\lambda$,

$$\langle T(v), w\rangle = \langle v, T(w)\rangle = \langle v, \lambda w\rangle = 0, \qquad \text{since } \lambda w \in E_\lambda.$$

  Hence, $T(v) \in E_\lambda^\perp$. QED

  In particular, T restricts to give a linear map $T_{|E_\lambda^\perp} : E_\lambda^\perp \to E_\lambda^\perp$, $v \mapsto T(v)$. This linear map is still self-adjoint. Hence, by induction, there exists a basis $B \subset E_\lambda^\perp$ consisting of eigenvectors of $T_{|E_\lambda^\perp}$ (i.e. eigenvectors of T).

  Choose any orthonormal basis $B' \subset E_\lambda$. Then, B' consists of eigenvectors of T.

  Lemma 2: $B \cup B'$ *is orthogonal*

  Proof of Lemma 2: This follows because $B \subset E_\lambda^\perp$ and $B' \subset E_\lambda$. QED

  In particular, $B \cup B'$ is linearly independent. Finally, since $V = E_\lambda + E_\lambda^\perp$, $B \cup B'$ spans V and is therefore a basis of V.

This completes the proof of the Spectral Theorem.

Remark: We will see similar proofs throughout this course.

Corollary: *Let $A \in M_k(\mathbb{C})$ be self-adjoint. Then, there exists a unitary matrix $U \in M_k(\mathbb{C})$ such that*

$$U^*AU = D \quad \text{is diagonal.}$$

Proof: Define $T : \mathbb{C}^k \to \mathbb{C}^k$, $v \mapsto Av$. Then, T is self-adjoint: $[T^*]_B = \overline{[T]}_B^t = \overline{A}^t = A = [T]_B \implies T = T^*$. Hence, by the Spectral Theorem, there exists an orthonormal basis $B \subset \mathbb{C}^k$ consisting of eigenvectors of T. Let U be the matrix having columns given by the vectors in B. Then, U is unitary so that $U^{-1} = U^*$. Hence, $U^*AU = D$ is a diagonal matrix. QED