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February 18: The Adjoint

Let $(V, \langle , \rangle)$ be an inner product space. Define the function

$$\alpha : V \to V^* = \{L : V \to \mathbb{C} \mid \ L \text{ linear}\}\ , \ v \mapsto \alpha_v$$

where

$$\alpha_v : V \to \mathbb{C}\ ,\ w \mapsto \langle w, v\rangle$$

• We showed last lecture that α is injective.

Claim: *α surjective*

Proof: Let $f : V \to \mathbb{C}$ be linear. We must find $v \in V$ such that

$$f(w) = \alpha_v(w) = \langle w, v\rangle, \quad \text{for every } w \in V$$

Choose an orthonormal basis $B = (v_1, \dots, v_k) \subset V$. Define

$$v = \sum_{i=1}^{k} \overline{f(v_i)} v_i$$

Then, for any $w \in V$,

$$\begin{aligned}\langle w, v\rangle &= \overline{\langle v, w\rangle} \\ &= \overline{\langle \sum_{i=1}^{k} \overline{f(v_i)} v_i, w\rangle} \\ &= \sum_{i=1}^{k} f(v_i)\overline{\langle v_i, w\rangle} \\ &= \sum_{i=1}^{k} f(v_i)\langle w, v_i\rangle\end{aligned}$$

We have already seen that

$$[w]_B = \begin{bmatrix}\langle w, v_1\rangle \\ \vdots \\ \langle w, v_k\rangle\end{bmatrix} \iff w = \sum_{i=1}^{k} \langle w, v_i\rangle v_i$$

Hence

$$f(w) = \sum_{i=1}^{k} f(v_i)\langle w, v_i\rangle \qquad \text{QED}$$

Remark: Observe that the proof for surjectivity relied on choosing a basis. It's a FACT that this reliance on an arbitrary choice is *unavoidable*: there's no natural bijection

$$V \to V^*,$$

all bijections require making some arbitrary choice.

In the language of **category theory**, the *dual space functor* is not an autoequivalence of the category of finite dimensional complex vector spaces.

Let $T : V \to V$ be linear. Then, for any $v \in V$, the composition

$$V \xrightarrow{T} V \xrightarrow{\alpha_v} \mathbb{C}$$

is linear i.e. $\alpha_v \circ T \in V^*$. The **adjoint of** T, is the function

$$T^* : V \to V$$

defined by the following rule: $T^*(v)$ is the *unique* element such that $\alpha_{T^*(v)} = \alpha_v \circ T$. The existence of such a unique element follows from the fact that α is a bijection.

In particular, for any $v, w \in V$, we have

$$\alpha_{T^*(v)}(w) = \alpha_v \circ T(w) \quad \iff \quad \langle w, T^*(v)\rangle = \langle T(w), v\rangle \qquad (!!!)$$

Property (!!!) is the defining property of T^*: if $S : V \to V$ is a linear map such that,

$$\langle w, S(v)\rangle = \langle T(w), v\rangle \qquad \text{for every } v, w \in V,$$

then $S = T^*$.

Proposition:

1. *T^* is linear.*
2. *$(S \circ T)^* = T^* \circ S^*$.*

Proof:

1. HW3
2. Let $v, w \in V$. Then,

$$\langle S(T(w)), v\rangle = \langle T(w), S^*(v)\rangle = \langle w, T^*(S^*(v))\rangle$$

Hence, $T^* \circ S^*$ satisfies (!!!) so that $(S \circ T)^* = T^* \circ S^*$. QED

Proposition: *Let $B = (v_1, \ldots, v_k) \subset V$ be an orthonormal basis, $T : V \to V$ linear. Then,*

$$[T^*]_B = \overline{[T]}_B^t$$

Here $\overline{A}^t$ denotes the conjugate transpose of the matrix A.

Proof: We have

$$[T^*(v_i)]_B = \begin{bmatrix} \langle T^*(v_i), v_1\rangle \\ \vdots \\ \langle T^*(v_i), v_k\rangle \end{bmatrix} = \begin{bmatrix} \overline{\langle v_1, T^*(v_i)\rangle} \\ \vdots \\ \overline{\langle v_k, T^*(v_i)\rangle} \end{bmatrix} = \begin{bmatrix} \overline{\langle T(v_1), v_i\rangle} \\ \vdots \\ \overline{\langle T(v_k), v_i\rangle} \end{bmatrix}$$

The transpose of this last column vector is precisely the i^{th} row of $\overline{[T]}_B$, the matrix obtained by conjugating all entries in $[T]_B$. QED

Definition:

- A linear map $T : V \to V$ is **unitary** if $T^{-1} = T^*$. This means, for every $v, w \in V$, that
$$\langle T(v), T(w)\rangle = \langle v, T^*(T(w))\rangle = \langle v, w\rangle$$
i.e. unitary maps are *inner product preserving*.

- Say T is **self-adjoint** if $T = T^*$. This means that, for every $v, w \in V$,
$$\langle T(v), w\rangle = \langle v, T(w)\rangle$$

- A $k \times k$ matrix A is **unitary** if $A^{-1} = \overline{A}^t$.

- A $k \times k$ matrix is **self-ajoint** if $A = \overline{A}^t$.

Remark: The identity
$$\overline{A}^t A = \mathbb{I}_k$$
implies that the columns of A are orthonormal. In particular, *A is unitary if and only if its columns are orthonormal.*

Proposition: *Let $T : V \to V$ be self-adjoint. Then, the eigenvalues of T are real.*

Proof: Let $\lambda \in \mathbb{C}$ be an eigenvalue of T, $v \in V$ an eigenvector associated to λ. Then,
$$\lambda\langle v, v\rangle = \langle T(v), v\rangle = \langle v, T(v)\rangle = \overline{\lambda}\langle v, v\rangle$$
Since eigenvectors are necessarily nonzero, we find $\lambda = \overline{\lambda}$ i.e. $\lambda \in \mathbb{R}$. QED