Contact: gwmelvin@colby.edu

February 15: Inner Product Spaces

Let V be a finite dimensional vector space over $\mathbb{C}$. An **inner product on** V is a function

$$\langle , \rangle : V \times V \to \mathbb{C}\ ,\ (u, v) \mapsto \langle u, v\rangle$$

such that, for any $u, v, w \in V$, $\lambda, \mu \in \mathbb{C}$

- $\langle \lambda u + \mu v, w\rangle = \lambda\langle u, w\rangle + \mu\langle v, w\rangle$
- $\langle u, v\rangle = \overline{\langle v, u\rangle}$
- $\langle u, u\rangle \geq 0$ and $\langle u, u\rangle = 0$ if and only if $u = 0_V$.

Exercise:

1. For any $v \in V$, $\langle 0_V, v\rangle = \langle v, 0_V\rangle = 0$, and
2. For any $u, v, w \in V$, $\lambda, \mu \in \mathbb{C}$, $\langle w, \lambda u + \mu v\rangle = \overline{\lambda}\langle w, u\rangle + \overline{\mu}\langle w, v\rangle$.

A vector space equipped with an inner product is called an. **inner product space**.

Define the **norm of** $v \in V$ to be

$$||v|| = \sqrt{\langle v, v\rangle}$$

Remark:

- An inner product is a generalisation of the dot product to complex vector spaces (also called a Hermitian inner product)
- An inner product is a tool we use to think about a complex vector space *geometrically*.

Example: $(\mathbb{C}^k, \langle , \rangle)$, the **standard inner product space**: where

$$\left\langle \begin{bmatrix} a_1 \\ \vdots \\ a_k \end{bmatrix}, \begin{bmatrix} b_1 \\ \vdots \\ v_k \end{bmatrix} \right\rangle = \sum_{i=1}^{k} a_i \overline{b}_i$$

We call this inner product the **standard inner product on** $\mathbb{C}^k$.

Definition: Let $(V, \langle , \rangle)$ be an inner product space, $v, w \in V$. Say v, w are **orthogonal** if $\langle v, w\rangle = 0$.

A subset $S \subset V$ is **orthogonal** if $\langle v, w\rangle = 0$, for all $v, w \in S$. If S is orthogonal and $||v|| = 1$, for every $v \in S$, we say that S is **orthonormal**.

Lemma: *Let $S \subset V$ be a collection of nonzero vectors. If S is orthogonal then S is linearly independent.*

Proof: Suppose we have a linear relation

$$a_1v_1 + \ldots + a_rv_r = 0_V$$

for $v_1, \ldots, v_r \in S$, some $a_1, \ldots, a_r \in \mathbb{C}$. Then, for each $j = 1, \ldots, r$,

$$0 = \langle \sum_{i=1}^{r} a_i v_i, v_j \rangle = \sum_{i=1}^{r} a_i \langle v_i, v_j \rangle = a_j \langle v_j, v_j \rangle, \quad \text{because } S \text{ is orthogonal.}$$

Since $v_j \neq 0_V$, $\langle v_j, v_j \rangle \neq 0 \implies a_j = 0$. QED

Fact: Any finite dimensional inner product space $(V, \langle , \rangle)$ contains an orthonormal basis. Moreover, if $B = (v_1, \ldots, v_k) \subset V$ is an orthonormal basis then

$$v = \langle v, v_1 \rangle v_1 + \ldots + \langle v, v_k \rangle v_k, \qquad \text{for every } v \in V,$$

$$\implies [v]_B = \begin{bmatrix} \langle v, v_1 \rangle \\ \vdots \\ \langle v, v_k \rangle \end{bmatrix} \in \mathbb{C}^k$$

Example:

1. The standard basis $S = (e_1, \ldots, e_k) \subset \mathbb{C}^k$ is orthonormal with respect to the standard inner product.
2. The basis $\left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix} \right)$ is orthonormal with respect to the standard inner product on $\mathbb{C}^2$.

The Adjoint

For any $v \in V$, we define a linear map

$$\alpha_v : V \to \mathbb{C}\ ,\ w \mapsto \langle w, v \rangle$$

In this way, we define a function

$$\alpha : V \to \mathrm{Hom}(V, \mathbb{C}) = \{L : V \to \mathbb{C} \mid \ L \text{ is linear}\}$$

We call $\mathrm{Hom}(V, \mathbb{C})$ the **dual space of** V, also denoted V^*.

Claim: *α is injective.*

Proof: If $\alpha(u) = \alpha(v)$ then, for all $w \in V$,

$$\alpha_u(w) = \alpha_v(w) \quad \implies \quad \langle w, u \rangle = \langle w, v \rangle \quad \implies \quad \langle w, u - v \rangle = 0$$

In particular, for $w = u - v$, we find

$$\langle u - v, u - v \rangle = 0 \quad \implies \quad u - v = 0_V \quad \implies \quad u = v. \quad \text{QED}$$