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Let $T : V \to V$ be a linear map, where V is a finite-dimensional vector space over $\mathbb{C}$. We say that $\lambda \in \mathbb{C}$ is an **eigenvalue of** T if there exists $v \in V$, $v \neq 0_V$, satisfying $T(v) = \lambda v$. We call such a nonzero vector an **eigenvector associated to** λ.

Remark: It's important to remember that eigenvalues/eigenvectors always come in pairs - you can't have one without the other.

Observe that

$$\begin{aligned} \lambda \text{ is an e-value of } T &\Leftrightarrow (T - \lambda \mathrm{id}_V)(v) = 0 \\ &\Leftrightarrow \ker(T - \lambda \mathrm{id}_V) \neq \{0_V \\ &\Leftrightarrow T - \lambda \mathrm{id}_V \text{ not invertible} \\ &\Leftrightarrow \det([T]_B - \lambda \mathrm{id}_V) \neq 0, \text{ for any basis } B \subset V. \end{aligned}$$

This observation leads to the following definition:

Definition/Proposition: Let $T : V \to V$ be a linear map, $B \subset V$ a basis. Then, the **characteristic polynomial of** T is

$$\chi_T(\lambda) = \det(T - \lambda \mathrm{id}_V) \in \mathbb{C}[\lambda]$$

The characteristic polynomial satisfies the following properties:

- $\deg \chi_T(\lambda) = \dim V$,
- the eigenvalues of T are precisely the roots of $\chi_T(\lambda)$,
- $\chi_T(\lambda)$ is independent of B.

The Fundamental Theorem of Algebra states: let $f(t) \in \mathbb{C}[t]$. Then, $f(t) = (t - c)p(t)$, for some $c \in \mathbb{C}$, $p(t) \in \mathbb{C}[t]$, $\deg p = \deg f - 1$ i.e. all complex polynomials admit a complex root. In particular, **any linear map $T : V \to V$ admits an eigenvalue.**

Definition: If $\lambda \in \mathbb{C}$ is an eigenvalue of T then define

$$E_\lambda = \ker(T - \lambda \mathrm{id}_V)$$

the $\lambda-$**eigenspace of** T. *Observe that E_λ is a subspace.*

Example: Consider the map

$$T : \mathbb{C}^3 \to \mathbb{C}^3 \; , \; \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} x_2 \\ x_3 \\ x_1 \end{bmatrix}$$

and the basis

$$B = \left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right)$$

Then,

$$A = [T]_B = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$\chi_T(\lambda) = \det(A - \lambda \mathbb{I}_3) = \det \begin{bmatrix} 1-\lambda & -1 & 0 \\ 1 & -\lambda & -1 \\ 0 & 0 & 1-\lambda \end{bmatrix} = 1 - \lambda^3$$

Using the standard basis $(S = (e_1, e_2, e_3) \subset \mathbb{C}^3$, we find

$$[T]_S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

In a similar way, we can show that

$$\det([T]_S - \lambda \mathbb{I}_3) = 1 - \lambda^3,$$

as claimed.

Hence, T admits three eigenvalues $1, \varepsilon, \varepsilon^2$, where $\varepsilon = e^{2\pi i/3}$. We have

$$E_1 = \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right), \quad E_\varepsilon = \operatorname{span}\left(\begin{bmatrix} 1 \\ \varepsilon \\ \varepsilon^2 \end{bmatrix}\right), \quad E_{\varepsilon^2} = \operatorname{span}\left(\begin{bmatrix} 1 \\ \varepsilon^2 \\ \varepsilon \end{bmatrix}\right)$$

DIAGONALISATION Let $T : V \to V$ be a linear map, V a finite-dimensional vector space over $\mathbb{C}$. Suppose we have a basis $B = (v_1, \dots, v_k) \subset V$ consisting of eigenvectors of T; say v_i has associated eigenvalue λ_i. Then,

$$[T]_B = \begin{bmatrix} [T(v_1)]_B & \cdots & [T(v_k)]_B \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \lambda_k \end{bmatrix} = D$$

is diagonal.

Let $C \subset V$ be any other basis of V, write $A = [T]_C$. Then,

$$[T(v)]_C = [T]_C [v]_C = A[v]_C, \qquad \text{for all } v \in V$$

$$\implies [T(v)]_B = P_{B \leftarrow C}[T(v)]_C = P_{B \leftarrow C} A [v]_C = P_{B \leftarrow C} A P_{C \leftarrow B} [v]_B$$

Therefore, by the AMAZING PROPERTY for $[T]_B$, we must have

$$D = [T]_B = P_{B \leftarrow C} A P_{C \leftarrow B}$$

That is, if $P = P_{C \leftarrow B}$ then

$$D = P^{-1} A P$$

Example: Let $T : \mathbb{C}^3 \to \mathbb{C}^3$ be as above, $S \subset \mathbb{C}^3$ the standard basis, $A = [T]_S$. If we let

$$B = \left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ \varepsilon \\ \varepsilon^2 \end{bmatrix}, \begin{bmatrix} 1 \\ \varepsilon^2 \\ \varepsilon \end{bmatrix} \right)$$

and

$$P = P_{S \leftarrow B} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \varepsilon & \varepsilon^2 \\ 1 & \varepsilon^2 & \varepsilon \end{bmatrix}$$

then

$$P^{-1}AP = D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon^2 \end{bmatrix}$$