Contact: gwmelvin@colby.edu

February 11: Linear Algebra Review

Recall: let $T : V \to W$ be a linear map, $B = (v_1, \dots, v_k) \subset V$, $C \subset W$ bases. Then, the matrix of T with respect to B, C is the *unique* $l \times k$ matrix $[T]_B^C$ satisfying

$$[T(v)]_C = [T]_B^C [v]_B, \qquad \text{for any } v \in V. \qquad \text{(AMAZING PROPERTY)}$$

• Why unique? Let A be a $l \times k$ matrix satisfying

$$[T(v)]_C = A[v]_B, \qquad \text{for any } v \in V.$$

Then, for each $i = 1, \dots, k$,

$$[T(v_i)]_C = A[v_i]_B$$

Observe that $[v_i]_B = e_i$ is the i^{th} standard basis vector. Hence, $A[v_i]_B = Ae_i$ is the i^{th} column of A. Therefore, the i^{th} column of any matrix satisfying the AMAZING PROPERTY must be equal to $[T(v_i)]_C$.

• Why must such a matrix exist? Let $C = (w_1, \dots, w_l)$. Then, if $v = \sum_{i=1}^{k} a_i v_i \in V$, we have

$$T(v) = \sum_{i=1}^{k} a_i T(v_i)$$

Hence, if

$$T(v_i) = \sum_{j=1}^{l} m_{ji} w_j, \qquad \text{for } i = 1, \dots, k$$

then

$$T(v) = \sum_{i=1}^{k} a_i T(v_i) = \sum_{i=1}^{k} \sum_{j=1}^{l} a_i m_{ji} w_j$$

That is,

$$[T(v)]_C = \begin{bmatrix} \sum_{i=1}^{k} a_i m_{1i} \\ \vdots \\ \sum_{i=1}^{k} a_i m_{li} \end{bmatrix}$$

It's straightforward to check that this column vector is the product of $[T]_B^C = [m_{ji}]_{j,i}$ (i.e. j indexes rows, i indexes columns) with $[v]_B = \begin{bmatrix} a_1 \\ \vdots \\ a_k \end{bmatrix}$.

Example: Let $V = W = \mathbb{C}^3$, $B = C = \left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right) = (v_1, v_2, v_3)$.

Define the linear map

$$T : \mathbb{C}^3 \to \mathbb{C}^3 , \quad \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \mapsto \begin{bmatrix} a_2 \\ a_3 \\ a_1 \end{bmatrix}$$

Then, to determine$[T]_B$ we must compute $[T(v_i)]_B$, for $i = 1, 2, 3$. Can do this as follows: consider the 3×6 matrix

$$[v_1\ v_2\ v_3\ T(v_1)\ T(v_2)\ T(v_3)] = \begin{bmatrix} 1 & 1 & 1 & 0 & -1 & 0 \\ - & -1 & 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

The right-hand 3×3 matrix is $[T]_B$!

$$[T]_B = \begin{bmatrix} -1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Remark: The above approach generalises to determine $[T]_B^C$ whenever $V = \mathbb{C}^k$, $W = \mathbb{C}^l$: consider the $l \times 2k$ matrix and row-reduce

$$[v_1\ \cdots\ v_k \mid T(v_1)\ \cdots\ T(v_k)] \sim [\mathbb{I}_3 \mid [T]_B^C$$

The right-hand $l \times k$ matrix is $[T]_B^C$.

Proposition: *If $T_1 : V \to W$ and $T_2 : W \to X$ are linear maps, $B \subset V, C \subset W, D \subset X$ are bases, then*

$$[T_2T_1]_B^D = [T_2]_C^D[T_1]_B^C$$

Proof: It suffices to show that the right-hand side of the claimed identity satisfies the AMAZING PROPERTY defining $[T_2T_1]_B^D$: for any $v \in V$,

$$[T_2(T_1(v))]_D = [T_2]_C^D[T_2]_B^C[v]_B$$

Let $v \in V$. Then,

$$\begin{aligned} [T_2]_C^D[T_1]_B^C[v]_B &= [T_2]_C^D[T_1(v)]_C, && \text{by AMAZING PROPERTY for } T_1 \\ &= [T_2(T_1(v))]_D, && \text{by AMAZING PROPERTY for } T_2 \end{aligned}$$

The result follows.

Given two bases $B = (v_1, \ldots, v_k), C \subset V$, define the **change of basis matrix from B to C** to be

$$P_{C \leftarrow B} = [\mathrm{id}_V]_B^C = [[v_1]_C \cdots [v_k]_C]$$

Proposition: *$P_{C \leftarrow B}$ is invertible with inverse $P_{B \leftarrow C}$*

Proof: We have

$$P_{C \leftarrow B}P_{B \leftarrow C} = [\mathrm{id}_V]_B^C[\mathrm{id}_V]_C^B = [\mathrm{id}_V \circ \mathrm{id}_V]_C = [\mathrm{id}_V]_C = \mathbb{I}_k$$

Remark: for any $v \in V$, we have

$$[v]_C = P_{C \leftarrow B}[v]_B$$

Example: Let $B = \left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right)$, $C = \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right)$.

Can compute $P_{C \leftarrow B}$ by row-reducing

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 & 2 & 1 \\ 0 & 1 & 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{bmatrix}$$

Then,

$$P_{C \leftarrow B} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$$