Representation Theory
Spring 2019
Contact: `gwmelvin@colby.edu`

Name: SOLUTION

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Throughout this test:

- G is a finite group, unless otherwise specified;
- U, V, W etc. are finite dimensional vector spaces over $\mathbb{C}$, unless otherwise specified
- $\langle , \rangle$ denotes an inner product,
- ρ, φ, ψ etc. are representations of G.

Definitions

1. Let (ρ, V) be a representation of G. Define what it means for a subspace $U \subseteq V$ to be a *subrepresentation*.

   Solution: For every $u \in U, g \in G$, $\rho_g(u) \in U$.

2. Define the *direct sum representation* $\rho \oplus \varphi$.

   Solution: Let (ρ, V), (φ, W) be the representations. The direct sum representation is the homomorphism
   $$\rho \oplus \varphi : G \to \mathrm{GL}(V \times W)$$
   where, for $(v, w) \in V \times W$, $g \in G$, $(\rho \oplus \varphi)_g(v, w) = (\rho_g(v), \varphi_g(w))$.

3. Define what it means for two representations ρ and φ to be *equivalent*.

   Solution: Let (ρ, V), (φ, W) be the representations. There exists an invertible linear map $T : V \to W$ such that $\varphi_g \circ T = T \circ \rho_g$, for every $g \in G$.

4. Define what it means for a nonzero representation ρ to be *irreducible*.

   Solution: If $U \subseteq V$ is a subrepresentation then either $U = \{0_V\}$ or $U = V$.

5. Let $(V, \langle , \rangle)$ be an inner product space. Define what it means for a representation $\rho : G \to \mathrm{GL}(V)$ to be *unitary*.

   Solution: For every $g \in G$, $\rho_g : V \to V$ is unitary i.e. $\langle \rho_g(u), \rho_g(v) \rangle = \langle u, v \rangle$, for every $u, v \in V$, $g \in G$.

True/False, Multiple Choice

1. Let (ρ, V) be a degree 2 representation. Suppose there exists a basis $B \subset V$ such that $[\rho_g]_B$ is upper-triangular, for every $g \in G$. Circle the true statement(s):

   (a) ρ is indecomposable.

   (b) There exists a degree 1 subrepresentation.

   (c) ρ is irreducible.

   (d) All of the above.

   Solution: (b) is the only correct statement.

2. Let $(\rho, V), (\varphi, U), (\psi, W)$ be nonzero representations of G. Suppose that $\rho \simeq \varphi \oplus \psi$. Circle the true statement(s):

   (a) for every basis $B \subset V$, the matrices $[\rho_g]_B$ are upper-triangular, for every $g \in G$.

   (b) for every basis $B \subset V$, the matrices $[\rho_g]_B$ are block diagonal, for every $g \in G$.

   (c) there exists a basis $B \subset V$ such that $[\rho_g]_B$ is block diagonal, for every $g \in G$.

   (d) there exists a subrepresentation $V_1 \subseteq V$ such that $\rho_{|V_1} \simeq \varphi$.

   (e) All of the above.

   Solution: (c), (d) are the only correct statements.

3. **True/False:**

   (a) Let $U \subseteq V$ be a proper subrepresentation, i.e. $U \neq \{0_V\}$ and $U \neq V$. Then, there exists a subrepresentation $W \subseteq V$ such that $V = U \oplus W$.

   Solution: True - Let (ρ, V) be the representation. Choose inner product with respect to which ρ is unitary; take $W = U^\perp$.

   (b) Let (ρ, V) be irreducible. Then, there exists a basis $B \subset V$ such that $[\rho_g]_B$ is lower-triangular, for every $g \in G$.

   Solution: False - if $B = (v_1, \ldots, v_k)$ then $[\rho_g]_B$ being lower-triangular means $\mathrm{span}(v_k)$ is a subrepresentation. If $\deg V > 1$ then this can't hold.

   (c) Let ρ be a representation of G on the inner product space $(V, \langle , \rangle)$. Then, ρ is unitary.

   Solution: False - we saw an example for $\mathbb{Z}/3\mathbb{Z}$ i.e. a degree 2 representation of $\mathbb{Z}/3\mathbb{Z}$ on $\mathbb{C}^2$ equipped with standard inner product that was not unitary.

   (d) Let (ρ, V) be an irreducible representation of degree > 1. Then, there does not exist a basis $B \subset V$ such that $[\rho_g]_B$ is diagonal, for every $g \in G$.

   Solution: True - if such a basis $B = (v_1, \ldots, v_k)$ did exist then $\mathrm{span}(v_1)$ is a subrepresentation.

   (e) Let (ρ, V) be a representation of G. Then,

   $$\varphi : G \to \mathrm{GL}(\mathbb{C}) \ , \ g \mapsto \det \rho_g$$

   defines a representation of G. Here, $\det \rho_g = \det[\rho_g]_B$, for some basis $B \subset V$: it's a fact that this definition is independent of the basis B.

   Solution: True - for every $g, h \in G$, $\varphi_{gh} = \det \rho_{gh} = \det(\rho_g \rho_h) = \det(\rho_g)\det(\rho_h) = \varphi_g \varphi_h$.

(f) Let (ρ, V) be a representation of the non-abelian group G. Then, $\varphi : G \to \mathrm{GL}(V)$, $g \mapsto \rho_g \circ \rho_g$ is a representation.

Solution: False - this one was way trickier than anticipated, apologies... Here's a counterexample: consider standard representation ρ of S_3 on $\mathbb{C}^3$ (this is not the usual notation...). Then, for every 2-cycle $g \in S_3$ we would have $\varphi_g = \rho_g \circ \rho_g = \rho_{g^2} = \rho_e = \mathbb{I}_3$. Note that $\varphi_{(123)} = \rho_{(123)^2} = \rho_{(132)}$. Then, noting that $(123)(23) = (12)$, we would have $\varphi_{(123)}\varphi_{(23)} = \rho_{(132)} \neq \rho_e = \varphi_{(12)} = \varphi_{(123)(23)}$. This shows φ is not a homomorphism.