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Inner Product Spaces

Throughout this note $(V, \langle , \rangle)$ is a finite dimensional inner product space over $\mathbb{C}$.

Gram-Schmidt Algorithm

Let $\{v_1, \dots, v_k\} \subset V$ be a linearly independent subset. The **Gram-Schmidt Algorithm** is a procedure to produce an orthogonal subset $\{u_1, \dots, u_k\} \subset V$ with the property that

$$\text{span}(v_1, \dots, v_i) = \text{span}(u_1, \dots, u_i), \qquad i = 1, \dots, k$$

Proceed as follows:

- Define $u_1 = v_1$.
- Having defined $u_1, \dots, u_{i-1}$, we define

$$u_i = v_i - \sum_{j=1}^{i-1} \frac{\langle v_i, u_j \rangle}{\langle u_j, u_j \rangle} u_j$$

We will now show that the set $\{u_1, \dots, u_k\}$ is orthogonal: first, observe that

$$\langle u_2, u_1 \rangle = \langle v_2 - \frac{\langle v_2, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1, u_1 \rangle = \langle v_2, u_1 \rangle - \frac{\langle v_2, u_1 \rangle}{\langle u_1, u_1 \rangle} \langle u_1, u_1 \rangle = 0.$$

Let $j > 1$. Suppose, for the purposes of induction, that $\langle u_j, u_i \rangle = 0$, for all $1 \leq i < j$. Then, for $1 \leq i < j + 1 \leq k$, we have

$$\langle u_{j+1}, u_i \rangle = \langle v_{j+1} - \sum_{r=1}^{j} \frac{\langle v_{j+1}, u_r \rangle}{\langle u_r, u_r \rangle} u_r, u_i \rangle = \langle v_{j+1}, u_i \rangle - \sum_{r=1}^{j} \frac{\langle v_{j+1}, u_r \rangle}{\langle u_r, u_r \rangle} \langle u_r, u_i \rangle$$

Since $1 \leq i < j + 1$, we have, by induction, $\langle u_r, u_i \rangle = 0$, for $1 \leq r \leq j$, $r \neq i$. Hence,

$$\langle u_{j+1}, u_i \rangle = \langle v_{j+1}, u_i \rangle - \frac{\langle v_{j+1}, u_i \rangle}{\langle u_i, u_i \rangle} \langle u_i, u_i \rangle = 0.$$

Therefore, using induction, we have shown that the set $\{u_1, \dots, u_k\}$ is orthogonal.

Theorem. *Let $(V, \langle , \rangle)$ be a finite dimensional inner product space. Then, there exists an orthonormal basis of V.*

Proof: Let $B' \subset V$ be any basis of V. Then, apply the Gram-Schmidt Algorithm to obtain an orthogonal set $B = \{u_1, \dots, u_k\} \subset V$. Hence, B is linearly independent. Since $|B| = |B'| = \dim V$, and B is linearly independent, B is a basis of V. Then, the set

$$\left\{ \frac{u_1}{||u_1||}, \dots, \frac{u_k}{||u_k||} \right\}$$

is an orthonormal basis of V.

Orthogonal Projections

Let $W \subset V$ be a subspace, $v \in V$. Define the **projection of** v **onto** W, denoted $\text{proj}_W(v)$, as follows: choose an orthonormal basis $(w_1, \ldots, w_r) \subset W$. Define

$$\text{proj}_W(v) = \sum_{i=1}^{r} \langle v, w_i \rangle w_i$$

By definition, $\text{proj}_W(v) \in W$.

Proposition 1. *$w' = \text{proj}_W(v) \in W$ is the unique element in W such that $||v - w|| > ||v - w'||$, for all $w \in W, w \neq w'$.*

Proof: Let $w' = \text{proj}_W(v) = \sum_{i=1}^{r} \langle v, w_i \rangle w_i$.

- **Claim 1.** $\langle v - \text{proj}_W(v), w \rangle = 0$, for any $w \in W$: indeed, let $w = \sum_{i=1}^{r} a_i w_i$. Then,

$$\begin{aligned}
\langle v - \text{proj}_W(v), w \rangle &= \langle v, w \rangle - \langle \text{proj}_W(v), w \rangle \\
&= \langle v, \sum_{i=1}^{r} a_i w_i \rangle - \langle \sum_{i=1}^{r} \langle v, w_i \rangle w_i, \sum_{i=1}^{r} a_i w_i \rangle \\
&= \sum_{i=1}^{r} \overline{a}_i \langle v, w_i \rangle - \sum_{i=1}^{r} \sum_{j=1}^{r} \langle v, w_i \rangle \overline{a}_j \langle w_i, w_j \rangle \\
&= \sum_{i=1}^{r} \overline{a}_i \langle v, w_i \rangle - \sum_{i=1}^{r} \langle v, w_i \rangle \overline{a}_i, \quad \text{since } \langle w_i, w_j \rangle = \begin{cases} 1, & i = j, \\ 0, & i \neq j \end{cases} \\
&= 0
\end{aligned}$$

- **Claim 2.** (Pythagoras Theorem) *Suppose $v, w \in V$ and $\langle v, w \rangle = 0$. Then,*

$$||v||^2 + ||w||^2 = ||v - w||^2$$

  This follows from a straightforward computation of $||v - w||^2 = \langle v - w, v - w \rangle$

Let $w \in W$, $w \neq w'$. Then, $w - w' \in W$ so that, by Claim 1, $\langle v - w', w - w' \rangle = 0$. By Claim 2, we find

$$||v - w'||^2 + ||w - w'||^2 = ||v - w||^2$$

. Hence, for any $w \neq w'$,

$$||v - w'||^2 < ||v - w||^2 \quad \Longrightarrow \quad ||v - w'|| < ||v - w||.$$

Remark. Proposition 1 implies that the projection of v onto W is independent of the choice of orthonormal basis B used to define it.

Orthogonal Complement

Let $W \subset V$ be a subset. Define the **orthogonal complement of** W to be

$$W^{\perp} = \{v \in V \mid \langle v, w \rangle = 0, \quad \text{for all } w \in W\}$$

Proposition 2.

1. $W^{\perp}$ is a subspace.
2. $W \cap W^{\perp} = \{0_V\}$.
3. $V = W + W^{\perp}$ i.e. for any $v \in V$, there exist $w \in W$ and $z \in W^{\perp}$ such that $v = w + z$. In fact, w, z are *unique*.

Proof:

1. Since $\langle 0_V, v \rangle = 0$, for any $v \in V$, $0_V \in W^{\perp}$. Now, let $u, v \in W^{\perp}$, $a, b \in \mathbb{C}$, $w \in W$. Then,

   $$\langle au + bv, w \rangle = a\langle u, w \rangle + b\langle v, w \rangle = 0 + 0 = 0$$

   Hence, $au + bv \in W^{\perp}$.

2. Let $v \in W \cap W^{\perp}$. Then,

   $$\langle v, v \rangle = 0 \qquad \Longrightarrow \qquad v = 0_V$$

3. Let $v \in V$. Then,

   $$v = \text{proj}_W(v) + (v - \text{proj}_W(v))$$

   Claim 1 shows that $v - \text{proj}_W(v) \in W^{\perp}$. Moreover, $\text{proj}_W(v) \in W$, by construction. Hence, $V = W + W^{\perp}$.

   Suppose $v = w + z = w' + z'$, with $w, w' \in W$, $z, z' \in W^{\perp}$. Then,

   $$w + z = w' + z' \qquad \Longrightarrow \qquad w - w' = z' - z \in W \cap W^{\perp} = \{0_V\}$$

   Hence, $w - w' = 0_V$ and $z' - z = 0_V$, so that $w = w', z = z'$ must be unique.