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Practice Midterm 2 Solution

Disclaimer: This Practice Midterm consists of problems of a similar difficulty as will be on the actual midterm. However, **problems on the actual midterm may or may not be quite different in nature, and may or may not focus on different course material**. However, the actual midterm will have a similar format: one true/false problem, one short-answer problem, three long-answer problems.

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1. True/False (no justification required)

   (a) The function $f(x) = |x+1|$ is not differentiable.

   (b) Let $f(x) = e^x + 5$, where $-1 \leq x \leq 10$. Then, $x = 10$ is a local maximum of $f(x)$.

   (c) Let $f(x)$ be differentiable everywhere. Suppose $f(0) = f(5) = 5$. Then, there exists $0 < u < 5$ such that $f'(u) = 5$.

   (d) If $f''(p) > 0$ then p is a local minimum.

   (e) Let $y = \sin(\cos(x))$. Then, $\frac{dy}{dx} = \sin(\cos(x)) + \cos(\sin(x))$.

   Solution:

   (a) True: $f'(-1)$ DNE

   (b) False: local maximum must be a critical point but $f(x)$ has no critical points

   (c) False: $f(x) = 5$ is a counterexample

   (d) False: let $f(x) = x^2$ then $f''(-1) = 2 > 0$ but $x = -1$ is not a local minimum.

   (e) False: $\frac{dy}{dx} = \cos(\cos(x))(-s \in (x))$.

2. (a) Let $g(t) = e^{\cos(t^3)}$. Determine $g'(t)$.

   (b) Let $f(x) = \frac{\sin(x^2)}{e^x}$. Determine $f'(x)$.

   Solution:

   (a) Use Chain Rule twice: $g'(t) = e^{\cos(t^3} \cdot (-\sin(t^3)) \cdot (3t^2)$.

   (b) $f(x) = \sin(x^2)e^{-x}$; use product rule and chain rule. We have $f'(x) = 2x\cos(x^2)e^{-x} - \sin(x^2)e^{-x}$

3. A piece of wire having length 10m is cut into two pieces. One piece is bent into a square and the other is bent into an equilateral triangle.

   (a) Let x be the side length of the square. Show that the total area A of the square and triangle can be expressed as

   $$A = x^2 + \frac{\sqrt{3}}{4}\left(\frac{10-4x}{3}\right)^2$$

   Hint: the area of an equilateral triangle having side length a is $\frac{\sqrt{3}}{4}a^2$.

   (b) Determine the side length of the square giving the largest possible area A.

   Solution:

(a) Since side length of the square is xm, its area is x^2. This gives the first term in A. To create the square we use $4x$m of the wire (the perimeter of the square is 4m). Thus, there's $10 - 4x$m of wire left to make the triangle. Each side of the triangle has length $\frac{10-4x}{3}$, and the given formula for the area of an equilateral triangle gives the remaining term in A.

(b) Compute

$$\frac{dA}{dx} = 2x + \frac{\sqrt{3}}{2}\left(\frac{10-4x}{3}\right)\cdot\left(-\frac{4}{3}\right) = \left(2 + \frac{8\sqrt{3}}{9}\right)x - \frac{20\sqrt{3}}{9}$$

Hence, $x = \frac{20\sqrt{3}}{18+8\sqrt{3}}$ is the unique critical point. Since $\frac{d^2A}{dx^2} = 2 + \frac{8\sqrt{3}}{9} > 0$, this critical point is a local minimum. Moreover, since the second derivative is always positive, A is concave up everywhere. Hence, the maximum must occur at the endpoints i.e. when either $x = 0$ or $x = 10/4 = 2.5$.

When $x = 0$, $A = \frac{\sqrt{3}}{4}\left(\frac{10}{3}\right)^2 = 4.811...$

When $x = 2.5$, $A = (2.5)^2 = 6.25$. Hence the maximal area corresponds to there being no triangle and the whole wire is bent into a square.

4. Let $g(x) = x\sqrt{4-x^2}$, $-1 \le x \le 2$. Find global maximum and global minimum for $g(x)$.

Solution: We have

$$g'(x) = \sqrt{4-x^2} - \frac{x^2}{\sqrt{4-x^2}}$$

Find critical points:

$$g'(x) = 0 \implies \sqrt{4-x^2} = \frac{x^2}{\sqrt{4-x^2}} \implies 4 - x^2 = x^2 \implies x = \pm\sqrt{2}$$

Since $-\sqrt{2} < -1$, we need only consider the critical point $x = \sqrt{2}$. Then, comparing with the values at the endpoints

x	-1	2	$\sqrt{2}$
$g(x)$	$-\sqrt{3}$	0	2

Hence, global minimum at $x = -1$ and global maximum at $x = \sqrt{2}$.

5. Let $f(x) = xe^{-x^2}$.

(a) Determine the local maxima/minima of $f(x)$.

(b) Determine the inflection points of $f(x)$.

(c) Using what you've found, sketch the graph of $f(x)$. (*You may find the following information useful:* $\lim_{x\to\pm\infty} f(x) = 0$.)

Solution:

(a) We compute

$$f'(x) = e^{-x^2} + xe^{-x^2}\cdot(-2x) = e^{-x^2}\left(1 - 2x^2\right)$$

Hence, critical points occur when $1 = 2x^2$ i.e. $x = \pm\frac{1}{\sqrt{2}}$. We use Second Derivative Test:

$$f''(x) = e^{-x^2}\cdot(-2x)(1-2x^2) - 4xe^{-x^2} = 2xe^{-x^2}\left(2x^2 - 3\right)$$

Since $f''(-1/\sqrt{2}) < 0$ and $f''(1/\sqrt{2}) > 0$, we find that $x = -1/\sqrt{2}$ is a local maximum while $x = 1/\sqrt{2}$ is a local minimum.

(b) Possible inflection points when $f''(x) = 0 \implies$ either $x = 0$ or $2x^2 = 3$ i.e. $x = \pm\sqrt{\frac{3}{2}}$. Let's check the sign of $f''(x)$ near these points:

	$x > -\sqrt{3/2}$	$-\sqrt{3/2} < x < 0$	$0 < x < \sqrt{3/2}$	$\sqrt{3/2} < x$
$f''(x)$	< 0	> 0	< 0	> 0

Since $f''(x)$ changes sign at each of these points they are all inflection points.

(c)