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Practice Midterm 1 **Solution**

Disclaimer: This Practice Midterm consists of problems of a similar difficulty as will be on the actual midterm. However, **problems on the actual midterm may or may not be quite different in nature, and may or may not focus on different course material**. However, the actual midterm will have a similar format: one true/false problem, one short-answer problem, three long-answer problems.

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1. True/False (no justification required)

   (a) If $\lim_{x\to a} f(x)$ exists then $f'(a)$ exists.

   (b) If $\log_2(x) = 5$ then $x = 32$.

   (c) If $f(5) = -1$ and $f(x)$ is continuous at $x = 5$ then $\lim_{x\to 5} f(x) = -1$.

   (d) $\frac{d}{dx}x^e = x^e$.

   (e) $\frac{d}{dx}\left(5xe^x + \pi^e \cos(x)\right) = 5e^x - e\pi^{e-1}\sin(x)$

   Solution:

   (a) False

   (b) True

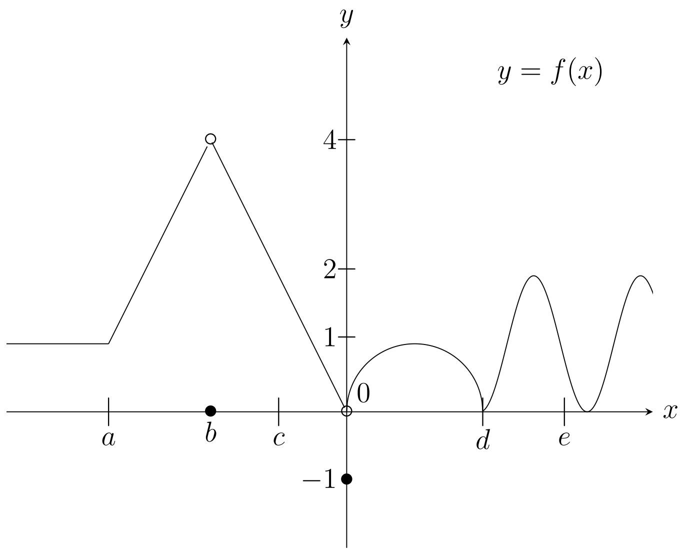
   (c) True

   (d) False

   (e) False

2. The graph of a function $f(x)$, defined for all x, is shown below. Using the given information, circle the true statements:

- $\lim_{x\to 0} f(x) = f(d)$ **T**
- $\lim_{x\to b^+} f(x) > f(0)$ **T**
- $\lim_{x\to a} f(x) = f(a)$ **T**
- $\lim_{x\to\infty} f(x) = 0$ **F**
- $\lim_{x\to d} f(x) = f(d)$ **T**
- $f'(a)$ DNE **T**
- $\lim_{x\to 0} f(x) = f'(a-1)$ **T**
- $\lim_{h\to 0} \dfrac{f(e+h)-f(e)}{h} < \lim_{x\to c} f(x)$ **T**
- $\lim_{x\to b} f(x) = 4$ **T**



3. Let k, l be constants.

(a) Using limit laws, or otherwise, determine

$$\lim_{x\to 1} \frac{x^2+kx-2}{x^2+5x-5}$$

Note: your answer will be in terms of k.

(b) Define

$$f(x) = \begin{cases} \frac{x^2+kx-2}{x^2+5x-5}, & x \neq 1 \\ l, & x = 1 \end{cases}$$

Determine all values of k, l so that $f(x)$ is continuous at $x = 1$. Be careful to justify why you know that $f(x)$ will continuous at $x = 1$ for these values of k, l.

Solution:

(a) *numerator:*

$$\lim_{x\to 1}(x^2+kx-2) \overset{LL1,LL2}{=} \lim_{x\to 1} x^2 + k\lim_{x\to 1} x - \lim_{x\to 1} 2 \overset{LL3,LL5,LL6}{=} (\lim_{x\to 1} x)^2 + k\cdot 1 - 2$$

$$\overset{LL6}{=} 1^2 + k - 2 = k - 1$$

denominator:

$$\lim_{x\to 1}(x^2+5x-5) \overset{LL1,LL2}{=} \lim_{x\to 1} x^2 + 5\lim_{x\to 1} x - \lim_{x\to 1} 5 \overset{LL3,LL5,LL6}{=} (\lim_{x\to 1} x)^2 + 5\cdot 1 - 5$$

$$\overset{LL6}{=} 1^2 + 5 - 5 = 1$$

Hence, using LL4,

$$\lim_{x\to 1} \frac{x^2+kx-2}{x^2+5x-5} = \frac{k-1}{1} = k-1$$

(b) In order that $f(x)$ is continuous at $x = 1$ we need

$$\lim_{x\to 1} f(x) = f(1) \implies k - 1 = f(1) = l$$

Here we note that, for $x \neq 1$, $f(x) = \frac{x^2+kx-2}{x^2+5x-5}$, so that $\lim_{x\to 1} f(x)$ has already been computed in (a).

4. Let $f(x) = \sqrt{2-x}$.

   (a) What is the domain of $f(x)$?

   (b) Determine the tangent line to the graph $y = f(x)$ at $(1, f(1))$.

   (c) Determine a formula for the inverse function $f^{-1}(x)$ and specify its domain.

   Solution:

   (a) We need $2 - x \geq 0$ i.e. $2 \geq x$.

   (b) We compute

   $$f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{\sqrt{1-h} - 1}{h}$$

   Multiplying the last expression by $\frac{\sqrt{1-h}+1}{\sqrt{1-h}+1}$, we get

   $$\lim_{h\to 0} \frac{1-h-1}{h(\sqrt{1-h}+1)} = \lim_{h\to 0} \frac{-1}{\sqrt{1-h}+1} = \frac{-1}{2}$$

   Hence, $f'(1) = -1/2$. Thus, using the point-slope formula, the tangent line is

   $$y - 1 = -\frac{1}{2}(x-1) \implies y = -\frac{x}{2} + \frac{3}{2}$$

   (c) Let $x = \sqrt{2-y}$: we want to write y in terms of x. Then, $f^{-1}(x) = y$.

   $$x = \sqrt{2-y} \implies x^2 = 2 - y \implies y = 2 - x^2$$

   Hence, $f^{-1}(x) = 2 - x^2$. The domain of $f^{-1}(x)$ is the range of $f(x)$ i.e. all non-negative real numbers.

5. Let $f(x) = \begin{cases} -x, & x < 0 \\ x^2, & x \geq 0 \end{cases}$

   (a) Using the limit definition of the derivative, explain why $f'(0)$ does not exist.

   (b) Draw the graph of the derivative $f'(x)$.

   Solution:

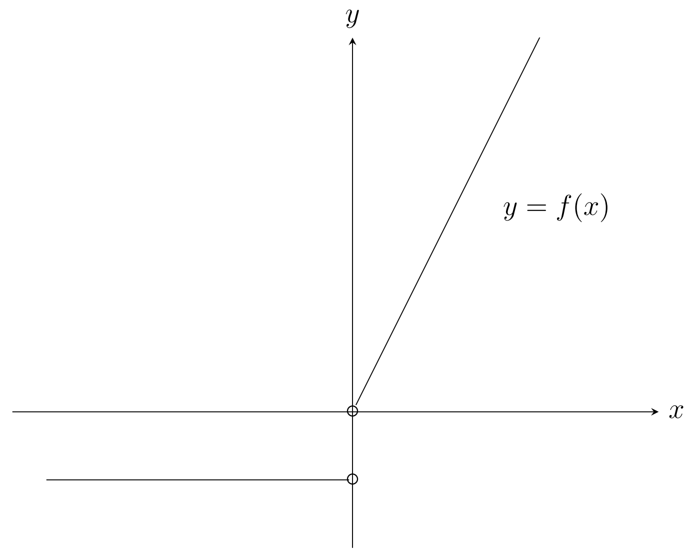
   (a) We compute

   $$\lim_{h\to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0^-} \frac{-h}{h} = \lim_{h\to 0^-} (-1) = -1$$

   Also,

   $$\lim_{h\to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0^+} \frac{h^2}{h} = \lim_{h\to 0^+} h = 0$$

   Since the two one-sided limits are not equal the two sided limit does not exist. Hence, $f'(0)$ does not exist.



(b)