Contact: gwmelvin@colby.edu

Practice Final: Solution

Disclaimer: This Practice Final consists of problems of a similar difficulty as will be on the actual final. However, **problems on the actual final may or may not be quite different in nature, and may or may not focus on different course material**. However, the actual final will have a similar format: one true/false problem, one short-answer problem, three long-answer problems.

---

1. True/False (no justification required)

   (a) The function $f(x) = |x|$ does not possess an antiderivative.

   (b) The formula $\int_0^x f''(x)dx = f'(x) - f'(0)$ holds.

   (c) If $F(x)$ is an antiderivative of $f(x)$ then $F'(x) = f(x)$.

   (d) $\int e^{x^2} dx = \frac{e^{x^2}}{2x} + C$

   (e) $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{\pi \sin(\pi k/n)}{n} = 2$

   Solution:

   (a) False: $f(x)$ is continuous and any continuous function possesses an antiderivative (SFTC).

   (b) True: $F(x) = f'(x)$ is an antiderivative of $f''(x)$.

   (c) True: this is the definition.

   (d) False: the derivative of the RHS is not equal to e^{x^2}.

   (e) True : the limit is the right hand sum R_n for $f(x) = \sin(x)$ on the interval $0 \le x \le \pi$. Therefore, $\lim_{n\to\infty} R_n = \int_0^\pi \sin(x)dx = 2$.

2. Compute the indefinite integral.

   (a)
   $$\int x\cos(\pi x + e)dx$$

   (b)
   $$\int \frac{1}{x\ln(x)}dx$$

   Solution:

   (a) We use integration by parts:

   $$u = x, \quad u' = 1, \qquad v' = \cos(\pi x + e), \quad v = \int \cos(\pi x + e)dx = \frac{1}{\pi}\sin(\pi x + e)$$

   Then, using $\int uv' = uv - \int u'v$,

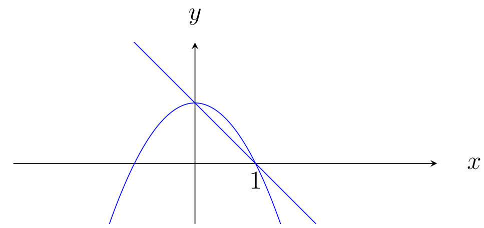
   $$\int x\cos(\pi x + e)dx = \frac{x}{\pi}\sin(\pi x + e) - \frac{1}{\pi}\int \sin(\pi x + e)dx$$
   $$= \frac{x}{\pi}\sin(\pi x + e) + \frac{1}{\pi^2}\cos(\pi x + e) + C$$

(b) We use method of u-substitution: let $u = \ln(x)$, so that $du = \frac{1}{x}dx$. Then,

$$\int \frac{1}{x\ln(x)}dx = \int \frac{1}{u}du = \ln|u| + C = \ln|\ln(x)| + C$$

3. Determine the area bounded between the curves $y = 1 - x$ and $y = 1 - x^2$.

Solution:



We compute

$$\int_0^1 1 - x^2 - \int_0^1 1 - x dx = \int_0^1 x - x^2 dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

4. Consider the function

$$f(x) = \int_0^{x^2} \sin(t^2)dt$$

(a) Compute $f'(x)$.

(b) Determine the equation of the tangent line to the curve $y = f(x)$ at $x = \pi$.

Solution:

(a) Let

$$g(x) = \int_0^x \sin(t^2)dt$$

Then, by SFTC, $g'(x) = \sin(x^2)$. We have

$$f(x) = g(x^2) \quad \implies \quad f'(x) = g'(x^2) \cdot (x^2)' = 2x\sin(x^4)$$

(b) The slope of the tangent line is

$$f'(\pi) = 2\pi\sin(\pi^4)$$

and the tangent line is

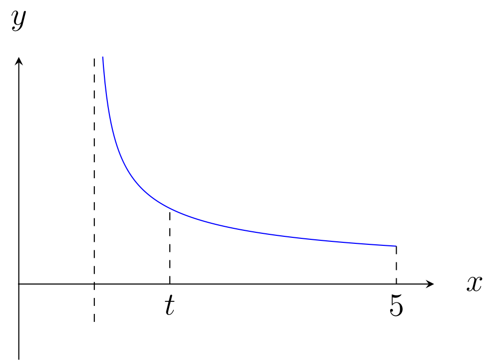
$$y - f(\pi) = 2\pi\sin(\pi^4)(x - \pi) \quad \implies \quad y = \int_0^{\pi^2} \sin(t^2)dt + 2\pi\sin(\pi^4)(x - \pi)$$

5. Let $f(x) = 1/\sqrt{x-1}$, defined when $x > 1$.

(a) Let $1 < t < 5$. Compute $A(t) = \displaystyle\int_t^5 f(x)dx$.

(b) Compute $\lim_{t\to 1^+} A(t)$.

(c) True/False: the area bounded below $y = f(x)$, $1 < x \leq 5$, is $\lim_{t\to 1^+} A(t)$. **Justify your answer**.



Solution:

(a)

$$A(t) = \int_t^5 (x-1)^{-1/2} dx = \left[2(x-1)^{1/2}\right]_t^5 = 2\left(2 - \sqrt{t-1}\right)$$

(b) Hence, since $2\left(2-\sqrt{t-1}\right)$ is defined and continuous at $t = 1$,

$$\lim_{t \to 1^+} 2(2 - \sqrt{t-1}) = 2(2 - \sqrt{1-1}) = 4$$

(c) True: $A(t)$ is the area below the the graph $y = f(x)$ on the interval $t \leq x \leq 5$. Therefore, the area beneath the graph $1 < x \leq$ will be the limit of this area as we let t get closer and closer, but not equal, to 1. This is precisely the definition of the limit $\lim\limits_{t \to 1^+} A(t)$.