Math 121B: Single-Variable Calculus
Spring 2019

Contact: gwmelvin@colby.edu

February 8 Summary

Supplementary References:
- *Calculus*, Hughes-Hallet et al, Sections 1.1, 1.3

Keywords: *function, increasing, decreasing, concave up/down, composition, inverse function*

Functions

• A **function** f is a rule that assigns to each input x exactly one output $y = f(x)$. The collection of all inputs is called the **domain of** f; the collection of all possible outputs is called the **range of** f.

What do we care about?

Representations of functions:

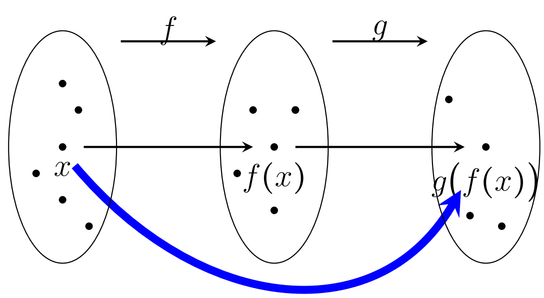
- in words
- table
- **graph** i.e. in (x, y)-plane plot the points $y = f(x)$
- **formula** e.g. $f(x) = x^2 + 3x - 5$

Descriptions of functions:

- increasing/decreasing
- concave up/down

• **Remark:** we focus on graph/formula representations; we will develop techniques to plot the graph given a formula.

• Let $f(x), g(x)$ be two functions with the property that the range of $f(x)$ is contained in the domain of $g(x)$ i.e. outputs of $f(x)$ are inputs of $g(x)$. We can form the **composition** $g(f(x))$:



- **Remark:** the notation can be confusing: $g(f(x))$ means

DO f FIRST AND THEN DO g

Example:

1. Let X be the collection of people in the classroom, Y the collection of all places on Earth, Z the collection of real numbers. Define

   $$f : X \to Y, \text{ people in classroom} \to \text{ birthplace}$$

   $g : Y \to Z$, place on Earth $\to$ distance from that place to Miller Library (in miles)

   The outputs of f are the inputs of g so we can form the composition $g(f(x))$: it is the function that assigns to each person in class the distance from their birthplace to the Miller Library.

2. Let $f(x) = x^2$, $g(x) = 2x^2 + 3x + 2$, where the domain for both functions is the collection of all real numbers. Then, outputs of f are inputs of g and we can form the composition:

   $$g(f(x)) = g(x^2) = 2(x^2)^2 + 3(x^2) + 2 = 2x^4 + 3x^2 + 2$$

- **WARNING:** Be careful! In the examples above:

1. $f(g(x))$ does not make sense: output of g is a number, while the input of f is a person. Can't form the composition in this instance.

2. $$f(g(x)) = f(2x^2+3x+2) = (2x^2+3x+2)^2 = 4x^4+12x^3+17x^2+12x+4 \neq g(f(x))!!$$

ORDER OF COMPOSITION IS IMPORTANT!

An interesting example:

$$f(x) = \frac{1}{1-x}, \quad \text{inputs all } x \neq 1$$

$$g(x) = 1 - \frac{1}{x}, \quad \text{inputs all } x \neq 0$$

Note: outputs of f are inputs of g: outputs of g are inputs of f $\Longrightarrow$ can form the compositions $f(g(x))$ and $(g(f(x))$.

$$f(g(x)) = f\left(1 - \frac{1}{x}\right) = \frac{1}{1-(1-1/x)} = \frac{1}{\frac{1}{x}} = x \qquad (*)$$

$$g(f(x) = g\left(\frac{1}{1-x}\right) = 1 - \frac{1}{\frac{1}{1-x}} = 1-(1-x) = x \qquad (**)$$

Remark:

1. We need to be careful to check that the output of f is an allowed input of g: we are saying that $f(x) = \frac{1}{1-x} \neq 0$, for any real number $x \neq 1$.

2. Remember what the notation $\frac{1}{a}$ means: $\frac{1}{a}$ is used to denote that unique real number b that has the property that $ab = 1$. Any number of the form $\frac{1}{a}$ can't equal 0: if it did then we would have $a \cdot 0 = 1$, which is ridiculous.