Math 121B: Single-Variable Calculus
Spring 2019

Contact: gwmelvin@colby.edu

February 18 Summary

Supplementary References:
- *Calculus*, Hughes-Hallet et al, Section 1.8

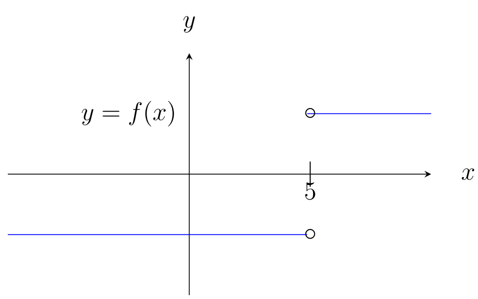
Keywords: *one-sided limits, Limit Laws*

Limits contd.

- Let $f(x)$ be a function. Write $\lim_{x \to a} f(x) = L$ if the *values of* $f(x)$ *approach* L *as* x *gets closer and closer,* **but not equal**, *to* a.

Example: Let $f(x) = \frac{|x-5|}{x-5}$, $x \neq 5$.

- If $x > 5$ then $x - 5 > 0$ and $|x - 5| = x - 5$. Hence, $f(x) = 1$ whenever $x > 5$.
- If $x < 5$ then $x - 5 < 0$ and $|x - 5| = -(x - 5)$. Hence, $f(x) = -1$ whenever $x < 5$.



- if x approaches 5 from the right then $f(x)$ approaches $L = 1$; we write

$$\lim_{x \to 5^+} f(x) = 1$$

- if x approaches 5 from the left then $f(x)$ approaches $L = -1$; we write

$$\lim_{x \to 5^-} f(x) = -1$$

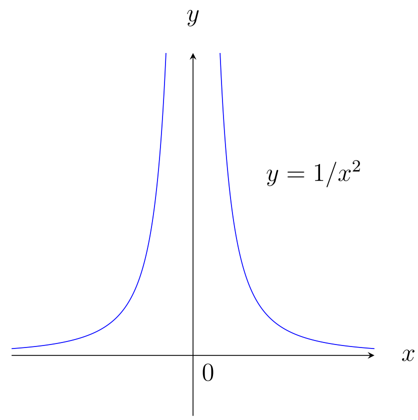
As x gets close to $a = 5$ there are two possibilities for $L \implies \lim_{x \to 5} f(x)$ **does not exist (DNE)**. Call $\lim_{x \to 5^\pm} f(x)$ **one-sided limits**.

- Even though the two-sided limit $\lim_{x \to 5} f(x)$ DNE, the two one-sided limits do exist.
- In general, $\lim_{x \to c} f(x)$ exists if both one-sided limits $\lim_{x \to c^\pm} f(x)$ exist and are equal.

- **Some further non-existent limits:**

1. Consider the function $f(x) = \frac{1}{x^2}$. Then, as x gets closer and closer to 0, $f(x)$ grows without bound i.e. there is no finite value L that $f(x)$ approaches as x approaches 0. Hence, $\lim_{x \to 0} f(x)$ DNE.

   In this very special case - when $f(x)$ grows without bound as x approaches 0 - we write $\lim_{x \to 0} f(x) = +\infty$. It's important to remember that this **limit does not exist**.



2. Let $f(x) = \frac{1}{x}$. Then, as x gets closer and closer to 0, with $x > 0$, $f(x)$ grows without bound in the positive direction. Hence, $\lim_{x \to 0^+} f(x)$ DNE and we write $\lim_{x \to 0^+} f(x) = +\infty$ to denote this behaviour.

   Similarly, as x gets closer and closer to 0, with $x < 0$, $f(x)$ grows without bound in the negative direction. Hence, $\lim_{x \to 0^-} f(x)$ DNE and we write $\lim_{x \to 0^+} f(x) = -\infty$ to denote this behaviour.

   In particular, neither one-sided limits exist.