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February 12 Summary

Supplementary References:
- *Calculus*, Hughes-Hallet et al, Sections 1.2, 1.4

Keywords: *Horizontal Line Test, exponential functions, logarithm*

Inverse functions; exponential functions

Horizontal Line Test: If the graph $y = f(x)$ has the property that any horizontal line $y = L$ intersects the graph at most once then $f(x)$ admits an inverse function $f^{-1}(x)$.

• We have the algebraic relationship

$$f^{-1}(f(x)) = x, \text{for every } x \text{ in the domain of } f(x),$$

$$f(f^{-1}(x)) = x, \text{for every } x \text{ in the domain of } f^{-1}(x).$$

Moreover,

- domain of $f(x)$ = codomain/range of $f^{-1}(x)$
- codomain/range of $f(x)$ = domain of $f^{-1}(x)$

• Let $f(x)$ be a function whose graph passes the HLT. **How to find a formula defining** $f^{-1}(x)$?

Consider $f(x) = \frac{1}{1-x}$, $x \neq 1$. Then, we must have

$$x = f^{-1}(f(x)) = f^{-1}\left(\frac{1}{1-x}\right)$$

i.e. if $y = f(x) = \frac{1}{1-x}$ then $f^{-1}(y) = x$. **Note:** the formula for $f^{-1}(y)$ should be in terms of $y \implies$ we need to solve the equation $f^{-1}(y) = x$ for y i.e. write x in terms of y.

$$\begin{aligned} y = \frac{1}{1-x} &\implies y(1-x) = 1 \\ &\implies 1 - x = \frac{1}{y} \\ &\implies x = 1 - \frac{1}{y} \end{aligned}$$

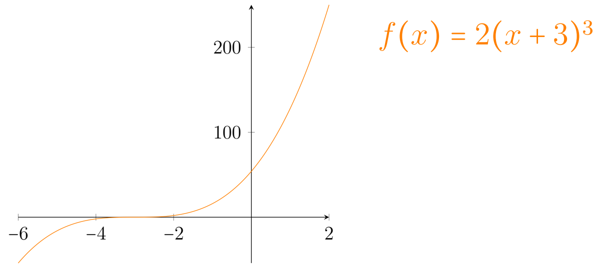
Hence, $f^{-1}(y) = 1 - \frac{1}{y}$.

To get a formula for $f^{-1}(x)$:

- set $x = f(y)$,

- solve for y in terms of x,
- $y = f^{-1}(x)$

Example: Let $f(x) = 2(x+3)^3$.



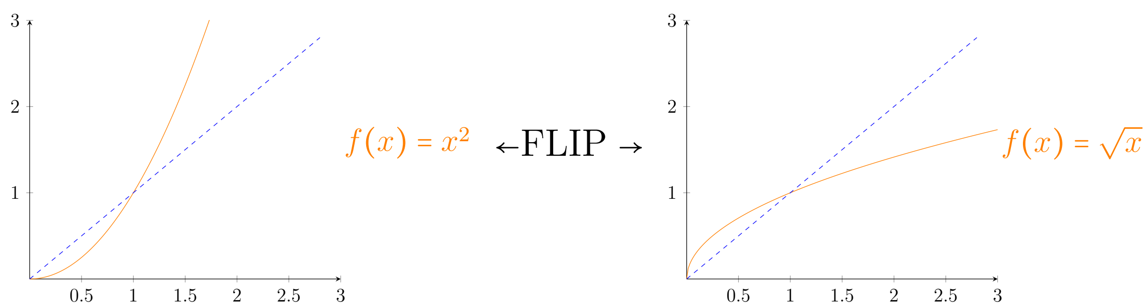
Using Desmos, we see that $y = f(x)$ passes HLT.

- Set $x = f(y) = 2(y+3)^3$
- Solve for y:

$$\begin{aligned} & x = 2(y+3)^3 \\ \implies & \frac{x}{2} = (y+3)^3 \\ \implies & \sqrt[3]{\frac{x}{2}} = y+3 \\ \implies & \sqrt[3]{\frac{x}{2}} - 3 = y \end{aligned}$$

- Hence, $f^{-1}(x) = y = \sqrt[3]{\frac{x}{2}} - 3$.

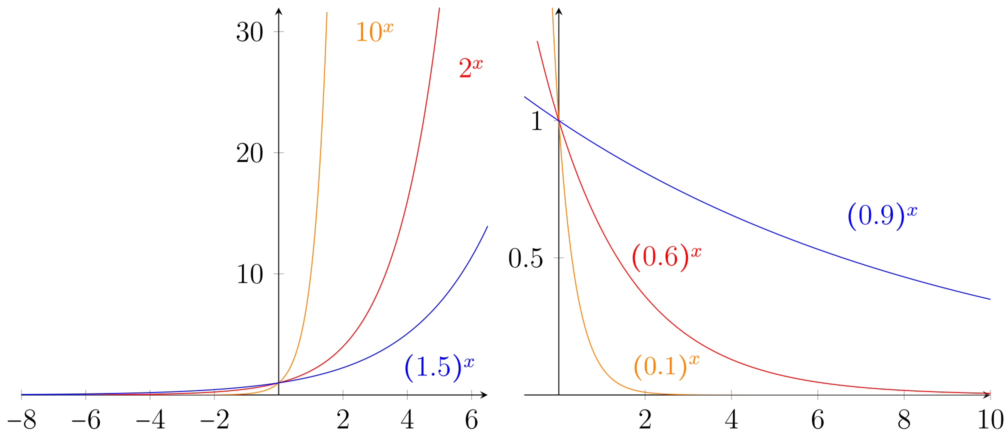
Remark: To get the graph of $f^{-1}(x)$ *flip* the graph $y = f(x)$ in the $y = x$ line:



Exponential functions:

- any function of the form $P(x) = Ka^x$, where K constant, $a > 0$, is called **exponential function with base a**
e.g. $P(x) = \frac{5^x}{2}$ is exponential with base 5.
- Domain of any exponential function is $\mathbb{R}$.
- **Remark:** Require $a > 0$ to ensure $P(x)$ is well-defined (e.g. to allow fractional inputs).
- If $a > 1$ then **exponential growth**; if $0 < a < 1$ then **exponential decay**. *Exponential functions model growth/decay of populations:*

- growth/decay of bacteria,
- compound interest,
- radioactive decay.



• The graph of $P(x) = a^x$ passes HLT $\implies$ $P(x)$ has an inverse. We call this inverse the *logarithm of base a*; denoted $\log_a(x)$.

• **Properties:**

$$a^{\log_a(x)} = x, \quad \log_a(a^x) = x$$

and

$$\log_a(x) = c \quad \Leftrightarrow \quad x = a^c$$

• domain of $\log_a(x)$ = range of a^x = positive real numbers.

• range of $\log_a(x)$ = domain of $a^x = \mathbb{R}$.