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February 11 Summary

Supplementary References:

- *Calculus*, Hughes-Hallet et al, Sections 1.2, 1.4

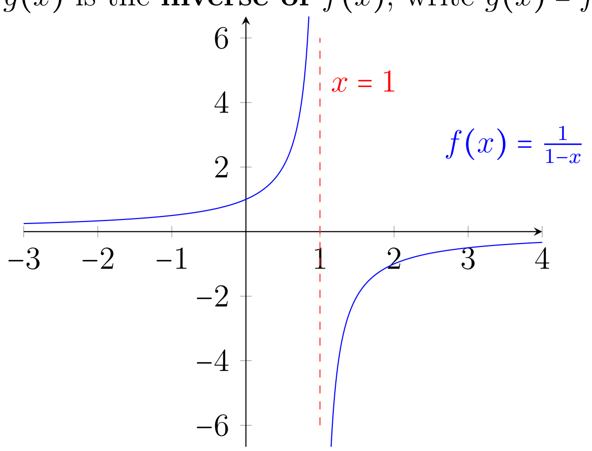
Keywords: *Horizontal Line Test, exponential functions*

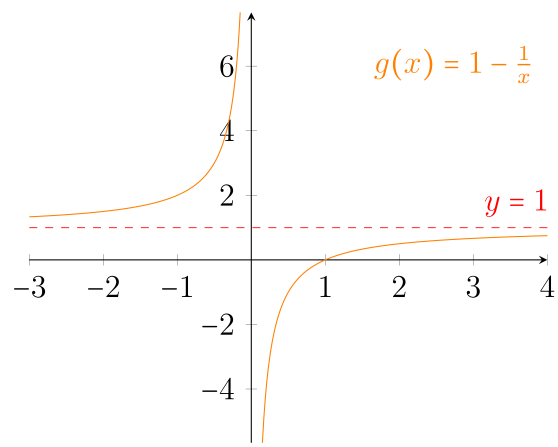
Horizontal Line Test; Exponential Functions

- **Notation:** $\mathbb{R}$ denotes the collection of all real numbers.
  - **Recall:** $f(x) = \frac{1}{1-x}$, $g(x) = 1 - \frac{1}{x}$. Then,

$$f(g(x)) = x, \quad \text{and} \quad g(f(x)) = x, \qquad (*)$$

i.e. $g(x)$ is the **inverse of** $f(x)$; write $g(x) = f^{-1}(x)$.

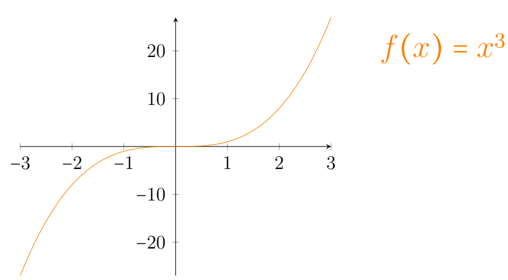




- Both graphs pass **Horizontal Line Test (HLT)**: *any horizontal line $y = L$ intersects the graph at most once.* This is graphical interpretation of $(*)$.
- HLT provides criterion to check if a function $f(x)$ admits an inverse: *if the graph $y = f(x)$ passes HLT then $f(x)$ has an inverse.*

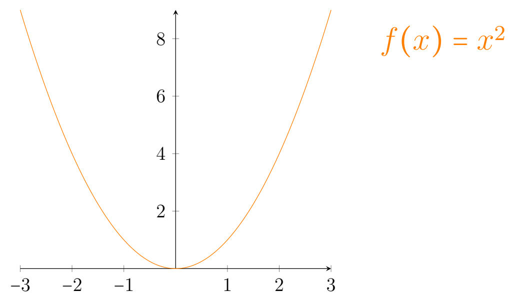
- **Example:**

1. $f(x) = x^3$, domain = $\mathbb{R}$.

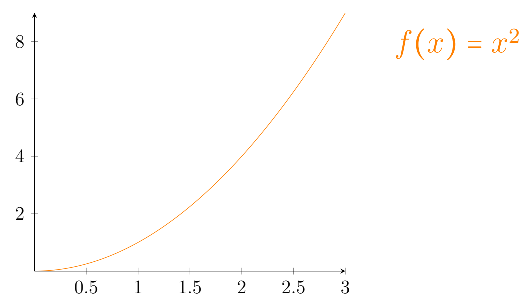


The graph $y = f(x)$ passes HLT $\implies$ $f(x)$ has an inverse $f^{-1}(x)$. What is it? f^{-1} is function that 'solves for x: i.e. if $y = f(x)$ then $f^{-1}(y) = x$ i.e. $f^{-1}(x^3) = x$. Hence, f^{-1} is the *cube root* function $f^{-1}(x) = \sqrt[3]{x} = x^{1/3}$.

2. $f(x) = x^2$, domain $= \mathbb{R}$.

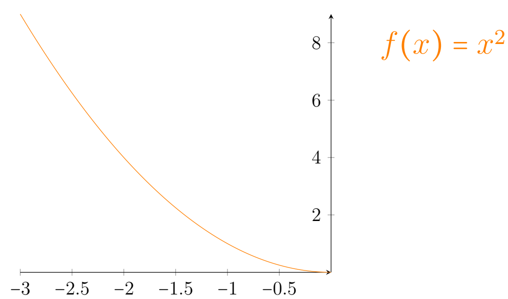


Does not pass HLT $\implies$ does not possess an inverse. **Can remedy the situation:** for example, restrict the domain to all nonnegative real numbers. Then, the graph of this new function is



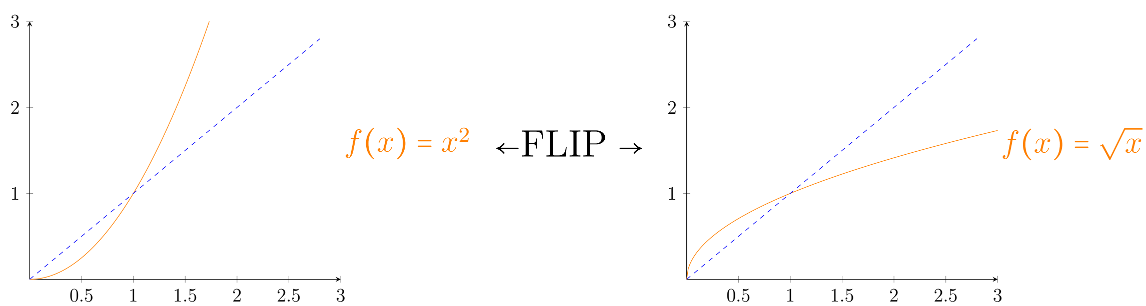
This function passes HLT: $f(x)$ has inverse $f^{-1}(x) = \sqrt{x}$.

- If we restrict the domain to nonpositive real numbers then the graph is



and $f^{-1}(x) = -\sqrt{x}$.

- To get the graph of $f^{-1}(x)$ *flip* the graph $y = f(x)$ in the $y = x$ line:



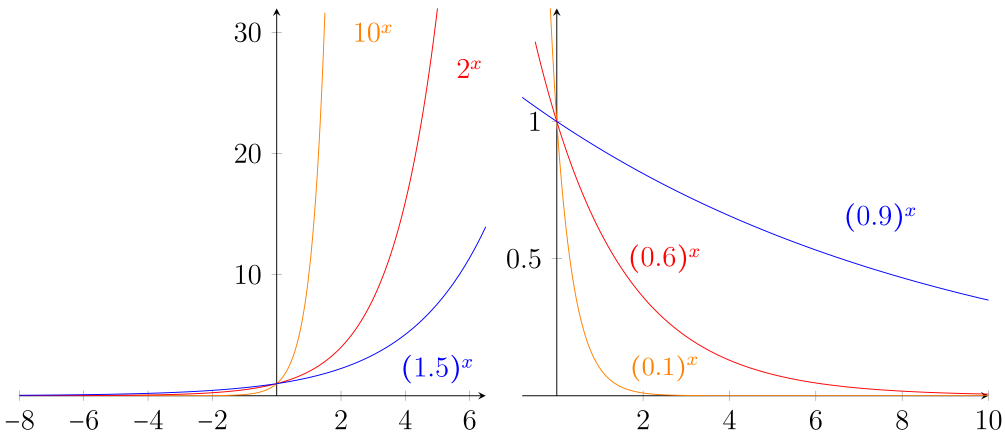
- To determine $f^{-1}(x)$ algebraically: set $f(y) = x$ and solve for y e.g. $f(x) = 2(x+3)^3$

$$x = f(y) = 2(y+3)^3 \quad \implies \quad y = \sqrt[3]{\frac{x}{2}} - 3 \quad \implies \quad f^{-1}(x) = \sqrt[3]{\frac{x}{2}} - 3$$

Exponential functions:

• any function of the form $P(x) = Ka^x$, where K constant, $a > 0$, is called **exponential function with base** a
e.g. $P(x) = \frac{5^x}{2}$ is exponential with base 5.
• Domain of any exponential function is $\mathbb{R}$.
• **Remark:** Require $a > 0$ to ensure $P(x)$ is well-defined (e.g. to allow fractional inputs).
• If $a > 1$ then **exponential growth**; if $0 < a < 1$ then **exponential decay**. *Exponential functions model growth/decay of populations:*

- growth/decay of bacteria,
- compound interest,
- radioactive decay.



• Euler's number $e = 2.71828\ldots$ plays important role in calculus. We will consider exponential functions with base e; exponential functions of the form $f(x) = e^{kx}$, where k is a constant.