MA434 Notes

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Section 3.1

Proposition-Definition

The following conditions on ring A are equivalent

(1) Every ideal I $\subset$ A is finitely generated.

(2) Every ascending chain of ideals eventually stabilizes.

(3) Every non-empty set of ideals of A has a maximal element.

If they hold, A is a Noetherian Ring.

Definitions

Finitely generated: Ideal I $\subset$ A is finitely generated if $\exists$ a_1, ...a_k $\in$ I such that I ={ $\alpha_1 a_1$, ...$\alpha_k a_k$ | α_i $\in$ A }.

Stabilizes: An ascending chain of ideals $I_1 \subset I_2 ... \subset I_M$ stabilizes if $\exists$ M such that $I_M = I_{M+1} = I_{M+2}....$

Proof

To prove all three definitions are equivalent, we will prove (1) $\implies$ (2), (2) $\implies$ (3), and (3) $\implies$ (1).

(1) $\implies$ (2) Proof

Assume I $\subset$ A is finitely generated.

To prove: $I_1 \subset I_2 ... \subset I_M = I_{M+1} = I_{M+2} ...$

Consider an ascending chain $I_1 \subset I_2 ... \cup I_j =$ I where I $\subset$ A.

Because I is finitely generated, it has generators $[f_1, ...f_k]$.

Each f_i is an element of some I_j. Notice, for example, if $f_7 \in I_5$, then $f_5 \in I_6 \subset I_7 ... \subset I_M$. Since there are finitely many generators, $\exists$ I_M such that $[f_1, ...f_k] \in I_M$. Then $I = I_M$ and the chain of ideals $I_1 \subset I_2 ... \subset I_M$ stabilizes at I_M.

(2) $\implies$ (3) Proof

Assume an ascending chain of ideals in A stabilizes.

$I_1 \subset I_2 ... \subset I_M = I_{M+1} = I_{M+2} ...$

To prove: Every non-empty set of ideals in A has a maximal element

There are two ways to prove this:

1st way: Zorn's Lemma- This makes the proof trivial so we won't do this.

2nd way: Create an ascending chain of ideals in A.

Ascending chain: $I_1 \subset I_2 ... \subset I_j$ If I_j is not maximal, then $\exists$ I_{j+1} such that $I_j \subset I_{j+1}$. Keep applying this logic until you eventually reach the maximal ideal I_M where $I_M = I_{M+1} = I_{M+2} ....$

(3) $\Longrightarrow$ (1) Proof

Assume every non-empty set of ideals in A has a maximal elemnet.

To prove: Every I $\subset$ A is finitely generated.

Let's consider the set F = { J $\subset$ I | J finitely generated subideal}

F has a maximal element J_o. There are two possible cases:

Case 1: J_o = I

Since J_o is finitely generated, I must be finitely generated.

Case 2: $J_o \neq$ I

Then $\exists$ $x \in$ I such that $x \notin J_o$. Then ¡ J_o, x¿ is larger than J_o, which is a contradiction.

$\therefore$ A is a Noetherian ring.

Section 3.2

Propositions

(i) Suppose that R is Noetherian, I $\subset$ R is an ideal, then R/I is Noetherian.

(ii) Suppose A is Noetherian and 0 $\notin$ S $\subset$ A, then B = A[S^{-1}] = { $\frac{a}{s}$ | $a \in$ A, s = 1 or products of $s_i \in$ S}

Proof of (i)

Let R be Noetherian, I $\subset$ R be an ideal.

To prove: $\forall$ ideal $\bar{B} \subset$ R/I, $\bar{B}$ is finitely generated (This is Proposition-Defnitiion (1) from Section 3.1).

R/I = { r + I | r $\in$ R}

Since $\bar{B} \subset$ R/I, $\forall$ b+I $\in$ B, $\forall$ r+I $\in$ R/I, br+I $\in \bar{B}$ where B = { b $\in$ R | b+I

$\in \bar{B}\}$. $\bar{B}$ has the form B/I where I $\subset$ B $\subset$ R and B is an ideal.

$\implies$ $\bar{B}$ is finitely generated.

Proof of (ii)

Assume A is Noetherian and 0 $\notin$ S $\subset$ A.

To prove: $\forall I_B \subset$ B is an ideal, I_B is finitely generated. The strategy is to write B in terms of the ideals in A.

Look at $I_B \cap$ A. $I_B \cap$ A is an ideal in A. $I_B \cap$ A absorbs products in A since I_B does and A does. Notice ae $\in I_B \cap$ A since a $\in$ A $\subset$ B and e $\in I_B \cap$ A.

A$[S^{-1}]$ = B

We put $[S^{-1}]$ next to the intersection and claim that this set is an ideal in B.

($I_B \cap$ A) $[S^{-1}]$ = { $\frac{e}{s}$ | e $\in I_B \cap$ A $\subset$ A, s = 1 or products of $s_i \in$ S }

This is an ideal in B. If we look at

$$\frac{e}{s} + \frac{e'}{s'} = \frac{es' + e's}{ss'}$$

the denominator is a product of elements in S and e's, es' $\in I_B \cap$ A because e' $\in I_B \cap$ A and s $\in$ S $\subset$ A.

We look at b $\in$ A $\subset$ B and e $\in I_B \cap$ A.

$$\frac{b}{s} + \frac{e}{s} = \frac{be}{s}$$

where $\frac{be}{s} \in I_B \cap$ A since b $\in$ A $\subset$ B and e $\in I_B \cap$ A.

Then $I_B \cap$ A $\subset$ B an is an ideal.

Claim: ($I_B \cap$ A) $[S^{-1}] = I_B$

Proof of claim: We will prove ($I_B \cap$ A) $[S^{-1}] \subset I_B$ and $I_B \subset$ ($I_B \cap$ A)$[S^{-1}]$.

Proof of $I_B \subset$ ($I_B \cap$ A)$[S^{-1}]$:

Let x $\in I_B$. x = $\frac{y}{s}$ where y $\in$ A $\subset$ B and s is the same as defined above.

x = $\frac{y}{s}$ = $\frac{xs}{s}$ where xs $\in$ A.

Notice x $\in I_B$ and s $\in$ S $\subset$ A $\subset$ B $\implies \frac{xs}{s} \in I_B \cap$ A) $[S^{-1}]$.

Proof of $(I_B \cap$ A)$[S^{-1}] \subset I_B$ is trivial. $\implies$ $(I_B \cap$ A) $[S^{-1}] = I_B$ and I_B is finitely generated $\implies$ B is Noetherian.

Now let's see what happens when we do this the other way. We start with an ideal in A, hit it with S^{-1}, and then intersect back with A.

$$I \subset A \to I[S^{-1}] \to I[S^{-1}] \cap A$$

where

$$A \to A[S^{-1}]$$

We won't answer what happens when we try the proposition the other way.

In the claim R Noetherian $\implies$ R/I Noetherian, we did not use anything about R being a domain or not.

In the claim A Noetherian $\implies$ A$[S^{-1}]$, our definitions do not work if A is not a domain.

Let's look at a new example. Suppose we have the affine plane. If you have a polynomial in two variables on the affine plane, you can use it to make a function. We want functions that we can compute everywhere but not at the origin. There are quotients of polynomials that can be computed everywhere outside of the origin, like $\frac{1}{x}$.

$$f \in k[x, y] \subset k(x, y)$$

Let's look at the set of rational functions for this.

R = { g $\in$ k(x, y) | g defines a function on $\mathbb{A}^{\nvDash}$ - (0, 0)}.

We're looking at rational functions where the denominator is nonzero at (0, 0). Let's allow these functions to be divided by any power of x.

$$R = \{\frac{f}{x^n} \mid f \in k(x, y)\}$$

These are functions that make sense off the y axis defined by x = 0.

In a sense, localization is moving away from certain points. We can also do the opposite and find functions that make sense on (0, 0) and some neighborhood but not in the whole plane.

$$R = \{\frac{f}{g} \mid g(0,0) \neq 0\}$$

Often localizations are much nicer than the ring itself. Here's an example in $\mathbb{Z}$:

$$\mathbb{Z} \subset \mathbb{Z}(p) = \{\frac{a}{b} \mid p \nmid b\} \subset \mathbb{Q}$$

where p is a prime. We know that there is a function from $\mathbb{Z} \to \mathbb{Z}/\mathrm{p}\mathbb{Z}$. The denominator of the set contains that functions that are invertible in $\mathbb{Z}/\mathrm{p}\mathbb{Z}$ so there is a function $\mathbb{Z}(\mathrm{p}) \to \mathbb{Z}/\mathrm{p}\mathbb{Z}$ where you send $\frac{1}{b}$ to tis inverse in $\mathbb{Z}/\mathrm{p}\mathbb{Z}$. This is the largest subring of $\mathbb{Q}$ where such a function exists. The ideals in $\mathbb{Z}$ are all generated by a single integer. How many of these are still interesting ideals in $\mathbb{Z}(\mathrm{p})$? If the ideals are generated by integers not divisible by p, then they are units, so we get the entire ring. The only elements that survive are ideals generated by powers of p.