SCRIBAL NOTE

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We conduct an informal investigation into the topological properties of algebraic curves over $\mathbb{C}$. To qualitatively characterize the degree of informality of our presentation, we quote Fernando himself: "Our aim is sightseeing, rather than a scientific expedition, so I will not worry too much if I fail to emphasize a subtle point here and there, nor if the theorems are less general than they could be, nor, in fact, if my readers do not learn all there is to know."[1] In the presentation to follow, we concern ourselves with only the nonsingular algebraic curves:

Convention. *Throughout our presentation, we consider only the nonsingular algebraic curves (whose degrees are to be specified) in the complex projective plane $\mathbb{P}^2(\mathbb{C})$; that is, suppose that an algebraic curve $\mathcal{C}$ of degree n is determined by the equation $F(X,Y,Z) = 0$, where $\deg F = n$, then we require that the system of equations*

$$\begin{cases} F = 0 \\ \frac{\partial F}{\partial X} = 0 \\ \frac{\partial F}{\partial Y} = 0 \\ \frac{\partial F}{\partial Z} = 0 \end{cases}$$

have no solutions in $\mathbb{C}^3 \backslash \{(0,0,0)\}$.

1. The Projective Line

We start with the simplest example of a projective object over $\mathbb{C}$: $\mathbb{P}^1(\mathbb{C})$, the projective line over $\mathbb{C}$. To visualize $\mathbb{P}^1(\mathbb{C})$, we observe that the object can be decomposed into a finite portion and a single point at infinity; that is,

$$\begin{aligned} \mathbb{P}^1(\mathbb{C}) &= \{[u:v] \mid u, v \in \mathbb{C};\ u,\ v \text{ are not both zero}\} \\ &= \{[u:1] \mid u \in \mathbb{C}\} \cup \{[1:0]\} \\ &\cong \mathbb{C} \cup \{\infty\}. \end{aligned}$$

We claim that this observation enables us to, topologically, see $\mathbb{C} \cup \{\infty\}$ as a sphere.

Consider the diagram shown in Figure 1.[2] Here, a sphere $\mathbb{S}^2$ is situated so that its origin O lies directly on the complex plane $\mathbb{C} \cong \mathbb{R}^2$. Letting N denote the north pole of the sphere, we see that each point $z \in \mathbb{C}$ is associated with a unique line passing through N and itself that intersects the sphere exactly once at Z. Conversely, given $Z \in \mathbb{S}^2 \backslash \{N\}$, the unique line passing through N and Z intersects the complex plane exactly once at z. Thus, we obtain a one-to-one correspondence between the points in $\mathbb{C}$ and all the points in $\mathbb{S}^2$ but N. We improve this further by associating the point at infinity with the north pole, thereby obtaining a bijection.

Date: Mar. 2, 2020.

[1]Source: *p*-adic Numbers: An Introduction (2nd Edition).

[2]Source: `mathematica.stackexchange.com/questions/23793/stereographic-projection`

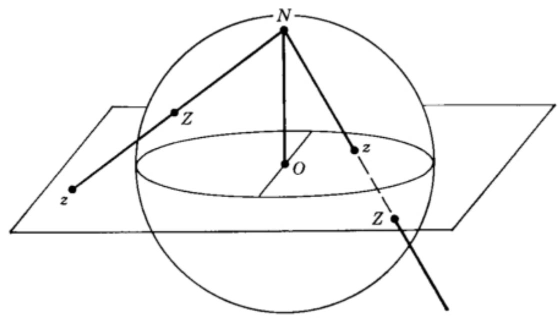


Figure 1. Stereographic projection.

Proposition 1. *The map*

$$\begin{aligned} \phi : \mathbb{S}^2 &\to \mathbb{C} \cup \{\infty\} \\ Z &\mapsto z \\ N &\mapsto \infty \end{aligned}$$

is a bijection. Consequently, $\mathbb{P}^1(\mathbb{C}) \cong \mathbb{C} \cup \{\infty\}$ *can be viewed, topologically, as a sphere.*

This technique is known more generally as stereographic projection. We state without proving two properties of this construction:

Proposition 2. *The inverse map* $\phi^{-1} : \mathbb{C} \cup \{\infty\} \to \mathbb{S}^2$ *maps any line on the complex plane, together with the point at infinity, into a circle on the sphere. Moreover,* ϕ^{-1} *preserves the angles of intersection.*

2. Lines and Conics

We are now prepared to state the topological characterizations of lines and conics in $\mathbb{P}^2(\mathbb{C})$.

Proposition 3. *Lines and conics in* $\mathbb{P}^2(\mathbb{C})$ *are homeomorphic to a sphere.*

Rather than provide a rigorous proof to Proposition 3, we will simply explain the reasoning behind its conclusion.

To start with, let $\mathcal{L} \subset \mathbb{P}^2(\mathbb{C})$ be a line. $\mathcal{L}$ is determined by the equation $aX + bY + cZ = 0$, where $a, b, c \in \mathbb{C}$ are not all zero. Via a suitable change of coordinates $T : (X, Y, Z) \mapsto (X', Y', Z')$, we may transform the equation into $X' = 0$. Equivalently, $\mathcal{L}$ is the set of points

$$\{[0 : Y' : Z'] \mid Y', Z' \in \mathbb{C};\ Y',\ Z' \text{ are not both zero}\}.$$

Noticing the natural bijection $\varphi_{\mathcal{L}} : \mathcal{L} \to \mathbb{P}^1(\mathbb{C})$ given by

$$[0 : Y' : Z'] \mapsto [Y' : Z'],$$

and resorting to Proposition 1, we conclude that $\mathcal{L}$ is homeomorphic to a sphere.

We now turn our attention to conics. Let $\mathcal{C}$ be a conic determined by some degree 2 equation $F(X, Y, Z) = 0$. Via a suitable change of coordinates, we may transform the equation into $Y'^2 = X'Z'$, which has the rational parameterization

$$\begin{cases} X' = U'^2 \\ Y' = U'V' \\ Z' = V'^2 \end{cases}$$

where $U', V' \in \mathbb{C}$ are not both zero. Again, we notice the natural bijection $\varphi_{\mathcal{C}} : \mathcal{C} \to \mathbb{P}^1(\mathbb{C})$ given by

$$[U'^2 : U'V' : V'^2] \mapsto [U' : V']$$

to conclude that $\mathcal{C}$ is homeomorphic to a sphere.

3. Cubics

We now focus on the topological characterization of the cubics.

Let $\mathcal{C} \in \mathbb{P}^2(\mathbb{C})$ be a cubic. We recall that $\mathcal{C}$ may be decomposed into a finite portion and a single point at infinity. Without loss of generality, we assume that the equation defining the finite portion of $\mathcal{C}$ is in the Weierstrauss form, i.e. it is determined by the algebraic equation $y^2 = f(x)$, where $f(x)$ is a degree 3 polynomial in x. Further, via a suitable change of coordinates if necessary, we can assume that the equation is $y^2 = x(x-1)(x-\lambda)$, where $\lambda \neq 0, 1$, and thus the corresponding projective equation of $\mathcal{C}$ is $Y^2Z = X(X-Z)(X-\lambda Z)$.

We observe that for each $x_0 \in \mathbb{C}\backslash\{0, 1, \lambda\}$, there are exactly two points on $\mathcal{C}$ with x-coordinate x_0, i.e. $(x_0, \pm\sqrt{x_0(x_0-1)(x_0-\lambda)})$. On the other hand, if $x_0 = 0, 1,$ or λ, then $(x_0, 0)$ is the only point on $\mathcal{C}$ with x-coordinate x_0. Moreover, we note that $[0 : 1 : 0]$ is the only point on $\mathcal{C}$ at infinity. Hence, we conclude that the map $\pi : \mathcal{C} \to \mathbb{P}^1(\mathbb{C})$ defined by

$$(x, y) \mapsto x$$
$$[0 : 1 : 0] \mapsto \infty$$

is two-to-one except at the four points $0, 1, \lambda,$ and $\infty \equiv [0 : 1 : 0]$.

In order to obtain a topological characterization of $\mathcal{C}$, we employ a strategy with which Riemann first yielded great success in his study of differential geometry. First, we use the fact that π defines a two-to-one (except for $0, 1, \lambda,$ and ∞) map from $\mathcal{C}$ to $\mathbb{P}^1(\mathbb{C})$ to obtain two copies of $\mathbb{P}^1(\mathbb{C})$. By Proposition 1, each copy of $\mathbb{P}^1(\mathbb{C})$ is homeomorphic to the sphere $\mathbb{S}^2$. To take into account the fact that π fails to be two-to-one at $0, 1, \lambda,$ and ∞, we make two cuts on each sphere along the path P_1 joining $\pi(0)$ and $\pi(1)$ and along a non-intersecting path P_2 joining $\pi(\lambda)$ and $\pi(\infty)$. Then, we glue the two copies of $\mathbb{S}^2$ together in such a way that the two copies of P_1 are pasted together and similarly are the two copies of P_2. Finally, we open up the slits formed as a result of our cutting along P_1 and P_2 respectively, and obtain a torus. See Figure 2.[3]

Hence, we conclude that:

Proposition 4. *Cubics in $\mathbb{P}^2(\mathbb{C})$ are homeomorphic to a torus.*

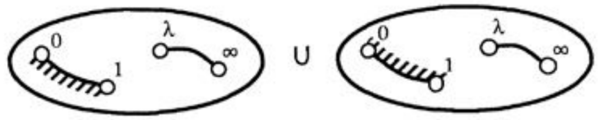
4. General Algebraic Curves

We now move on to discuss the topological characterization of general algebraic curves over $\mathbb{C}$, thought of as objects in $\mathbb{P}^2(\mathbb{C})$.

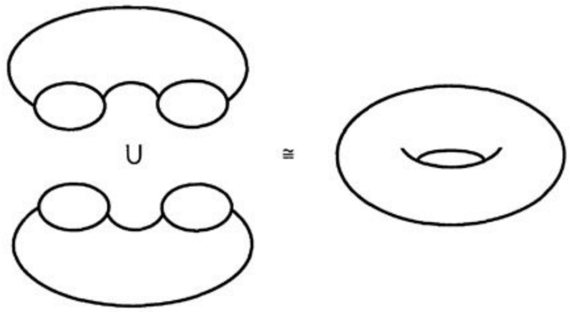
Let $\mathcal{C} \subset \mathbb{P}^2(\mathbb{C})$ be any algebraic curve over $\mathbb{C}$, and suppose that $\mathcal{C}$ is defined by the equation $F(X, Y, Z) = 0$. We note that the equation $F(X, Y, Z) = 0$ defines a closed subset of $\mathbb{P}^2(\mathbb{C})$. Thus, by accepting the fact that $\mathbb{P}^2(\mathbb{C})$ is compact, we conclude that $\mathcal{C}$ is also compact.

It is known that for every algebraic curve over $\mathbb{C}$, there is an associated complex surface that is compact, orientable over $\mathbb{C}$, and has a one-dimensional (over $\mathbb{C}$) tangent plane at each point. Such surfaces have been completely classified by their genus, g, which, intuitively, is

[3]Source: Undergraduate Algebraic Geometry.



(A) Cutting



(B) Pasting

FIGURE 2. Visualizing a cubic.

the number of "holes on the surface." For example, a sphere has $g = 0$, while a torus has $g = 1$. See Figure 3.[4]

FIGURE 3. The classification of orientable surfaces (2-manifolds).

It turns out that we may obtain a lot of information about the algebraic curve $\mathcal{C}$ itself from the genus of its associated surface alone. In our enumeration of some examples of these known properties, we follow the conventional trichotomy: $g = 0$, $g = 1$, and $g \geq 2$.

4.1. $g = 0$. If the associated surface of $\mathcal{C} \subset \mathbb{P}^2(\mathbb{C})$ is homeomorphic to $\mathbb{S}^2$, then

(1) $\mathcal{C}$ is either a line or a conic;
(2) $\mathcal{C}$ has a rational parameterization;
(3) the fundamental group of $\mathcal{C}$ is simply connected, i.e. $\pi_1(\mathcal{C}) = \{1\}$;
(4) the surface admits a metric with constant positive curvature;
(5) $\mathcal{C}(\mathbb{Q})$, the set of rational points on $\mathcal{C}$, is either empty or isomorphic to $\mathbb{P}^1(\mathbb{Q})$;
(6) all such $\mathcal{C}$'s are isomorphic to each other;
(7) the automorphisms of $\mathcal{C}$ are the projective transformations, e.g. the Mobius transformations $z \mapsto \frac{az+b}{cz+d}$ with $ad - bc \neq 0$, and these automorphisms form a 3-dimensional group.

[4]Source: `www.map.mpim-bonn.mpg.de/2-manifolds`

4.2. **$g = 1$.** If the associated surface of $\mathcal{C}$ is homeomorphic to $\mathbb{T}^2$, then

(1) $\mathcal{C}$ is a smooth cubic;
(2) $\pi_1(\mathcal{C}) \cong \mathbb{Z} \oplus \mathbb{Z}$;
(3) the surface admits a metric with zero curvature;
(4) $\mathcal{C}(\mathbb{Q})$ forms a finitely generated Abelian group;
(5) $\mathcal{C}$ belongs to a one-dimensional isomorphism class;
(6) the group of automorphisms of $\mathcal{C}$ is isomorphic to the direct product of the group of translations in the group law and some finite group;
(7) there exists a group homomorphism $\mathbb{C} \to \mathcal{C}$ whose kernel is isomorphic to $\mathbb{Z} \times \mathbb{Z}\tau$, where $\tau \notin \mathbb{R}$.

4.3. **$g \geq 2$.** If the associated surface of $\mathcal{C}$ has genus $g \geq 2$, then

(1) $\mathcal{C}$ is determined by some algebraic equation with degree higher than 3;
(2) $\pi_1(\mathcal{C})$ is close to a free group;
(3) the surface admits a metric with constant negative curvature;
(4) $\mathcal{C}(\mathbb{Q})$ is finite;
(5) $\mathcal{C}$ belongs to a $(3g-3)$-dimensional isomorphism class;
(6) the automorphisms of $\mathcal{C}$ form a finite group.

We note that it is an open problem to determine the size of $\mathcal{C}(\mathbb{Q})$.