Problem Day 1

Ethan Pullen

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1 Problem 1.6

Let k be a field with at least 4 elements, and $C : (XZ = Y^2)$ prove that if $Q(X, Y, Z)$ is a quadratic form which vanishes on C then $Q = \lambda(XZ - Y^2)$

1.1 Proof

Let $Q(X, Y, Z)$ be a quadratic such that it vanishes on $C : (XZ = Y^2)$. We can write out the equation for $Q = aX^2 + 2bXY + cY^2 + 2dXZ + 2eYZ + fZ^2$. We can now move the $2dXZ$ with part of cY^2 to achieve,

$$Q = 2d(XZ - Y^2) + aX^2 + 2bXY + (c - 2d)Y^2 + 2eYZ + fZ^2$$

Since $C : (XZ = Y^2)$, $Q[0 : 0 : 1] = 0 = f$, so $f = 0$. We can now rewrite Q,

$$Q = 2d(XZ - Y^2) + aX^2 + 2bXY + (c - 2d)Y^2 + 2eYZ$$

we can use the points $[1 : y : y^2] \in C$. Since Q vanishes over C, $Q[1 : y : y^2] = 0 = a + 2by + (c - 2d)y^2 + 2ey^3$. We are left with a cubic, but since k is a field with at least 4 elements, there are at least 4 zeroes of our cubic. The only way for that to happen is for all of the coefficients to be 0. We can now write,

$$Q = 2d(XZ - Y^2) + 0 * X^2 + 2bXY + 0 * Y^2 + 0 * YZ = 2d(XZ - Y^2)$$

Thus, $Q = \lambda(XZ = Y^2)$ where $\lambda = 2d$.

2 Problem 1.7

In R^3, consider the two planes $A : (Z = 1)$ and $B : (X = 1)$; a line through 0 meeting A in $(x, y, 1)$ meets B in $(1, \frac{y}{x}, \frac{1}{x})$. Consider the map $\phi : A \to B$ defined by $(x, y) \mapsto (y' = \frac{y}{x}, z' = \frac{1}{x})$; what is the image under ϕ of

2.1 the line $ax = y + b$

The line $ax = y+b$ is a pencil of parallel lines each with slope a. We will start by looking at where ϕ sends a line. Our mapping sends $(x, y) \mapsto (y' = \frac{y}{x}, z' = \frac{1}{x})$. We can solve our equation of a line for $\frac{y}{x}$ by subtracting b and dividing by x, $\frac{y}{x} = a + \frac{b}{x}$. So, $\phi : ax = y + b \mapsto (1, a - \frac{b}{x}, \frac{1}{x})$. $(1, a - \frac{b}{x}, \frac{1}{x})$ is a line with the equation $y = a - bz$. Since b can vary, our group of parallel lines in A are now a pencil of lines on the $x = 1$ plane with varying slopes that all go through $(1, a, 0)$.

2.2 circles $(x - 1)^2 + y^2 = c$ for variable c

We break this into 3 cases on c.
Case $c > 1$:
If $c > 1$, ϕ sends our circle equation to $(1, \frac{\pm\sqrt{c-(x-1)^2}}{x}, \frac{1}{x})$. We will let $\alpha = c - 1 > 0$, so we have $(1, \pm\sqrt{\frac{\alpha}{x^2} + \frac{2}{x} - 1}, \frac{1}{x})$. We can now write an equation, $y = \pm\sqrt{\alpha z^2 + 2z + 1}$, so $y^2 - \alpha z^2 - 2z + 1 = 0$. This is the equation of a hyperbola since α is positive.
Case $c = 1$:
If $c = 1$, ϕ sends our circle equation to $(1, \frac{\pm\sqrt{c-(x-1)^2}}{x}, \frac{1}{x}) = (1, \frac{\pm\sqrt{1-(x-1)^2}}{x}, \frac{1}{x}) = (1, \frac{\sqrt{2x-x^2}}{x}, \frac{1}{x}) = (1, \pm\sqrt{\frac{2}{x} - 1}, \frac{1}{x})$. So, $y = \pm\sqrt{2z - 1}$ giving us a parabola $y^2 - 2z + 1 = 0$.
Case $c < 1$:
If $c < 1$, ϕ sends our circle equation to $(1, \frac{\pm\sqrt{c-(x-1)^2}}{x}, \frac{1}{x})$. We will let $\alpha = -1 + c > 0$, so we have $(1, \pm\sqrt{-\frac{\alpha}{x^2} + \frac{2}{x} - 1}, \frac{1}{x})$. We can now write an equation, $y = \pm\sqrt{-\alpha z^2 + 2z + 1}$, so $y^2 + \alpha z^2 - 2z + 1 = 0$. This is the equation of an ellipse since α is positive.

3 Problem 1.8

3.1 Let $P_1, P_2, P_3, P_4 \in P^2$ with no 3 collinear. Prove that there is a unique coordinate system in which the 4 points are (1, 0, 0), (0, 1, 0), (0, 0, 1) and (1,1,1).

We want to define a linear transformation M such that:
$(1, 0, 0) \mapsto P_1$
$(0, 1, 0) \mapsto P_2$
$(0, 0, 1) \mapsto P_3$
$(1, 1, 1) \mapsto P_4$ Since $P_1, P_2, P_3, P_4 \in P^2$ we are allowed to scale them so that $P_1 + P_2 + P_3 = P_4$ No 3 points are collinear, so P_1, P_2, P_3 span R^3 which means there is some α, β, γ with $\alpha P_1 + \beta P_2 + \gamma P_3 = P_4$. So we want M to map each standard unit to its scaled version in P^2.
$M(1, 0, 0) = \alpha P_1$, $M(0, 1, 0) = \beta P_2$, $M(0, 0, 1) = \gamma P_3$
This will force $M(1, 1, 1) = P_4$. Thus our transformation to the coordinate system is simply M^{-1}.

3.2 Find all conics passing through $P_1 ... P_5$, where $P_5 = (x, y, z)$ is some other point

Let C be our conic, $C : aX^2 + 2bXY + cY^2 + 2dXZ + 2eYZ + fZ^2 = 0$. Since P_1, P_2, P_3 are on the curve, the points $(1, 0, 0), (0, 1, 0), (0, 0, 1)$ are zeroes on the conic, this means $a, c, f = 0$. Now we have $2bXY + 2dXZ + 2eYZ = 0$. P_4 is also on the curve, so $(1, 1, 1)$ is also a zero, thus $b + d + e = 0$. Using P_5, $bxy + dxy + eyz = 0$. We now have 2 equations for 3 variables, which means we have one solution in P^2.

3.3 Corollary 1.10

If $P_1 ... P_5 \in P^2$ are distinct points such that no 4 are collinear, then there exists at most one conic through $P_1 ... P_5$
We have shown that there is a unique way to move our coordinates to our new space and also that each time we add a fifth point, we define a single conic. Suppose there were 2 conics that go through all 5 points. This means there are 2 distinct ways to convert our coordinates, and the transformation would not be unique, thus it is impossible for 2 conics to exist.

4 Problem 1.10 and 1.11

Two forms on an algebraically closed field share a root if and only if Sylvester's Determinant is 0.

$$\begin{bmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \dots & \alpha_n & 0 & 0 & \dots & 0 \\ 0 & \alpha_0 & \alpha_1 & \alpha_2 & \dots & \alpha_n & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & \alpha_0 & \alpha_1 & \alpha_2 & \dots & \alpha_n \\ \beta_0 & \beta_1 & \beta_2 & \dots & \beta_m & 0 & 0 & \dots & 0 \\ 0 & \beta_0 & \alpha_1 & \beta_2 & \dots & \beta_m & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & \beta_0 & \beta_1 & \beta_2 & \dots & \beta_m \end{bmatrix}$$

4.1 Generalized Proof

Let A be an n degree form and B be an m degree form. We will assume A and B share a root $(\alpha : \gamma)$. There will be m variations of A ($U^x V^y A$ with $x + y = m$) and n variations of B ($U^x V^y B$ with $x + y = n$). Since A and B both have root $(\alpha : \gamma)$, any multiple of A and B will also have this root. Also, since all rows of Sylvester's Determinant are variations of A and B, all linear combinations will also share the root. Let $(\theta : \phi) \neq (\alpha : \gamma)$. Consider K, the $m + n$ degree form whose only root is $(\theta : \phi)$. Since this form doesn't share a root with A and B, it is not possible to create a linear combination to create K. This means the matrix form of Sylvester's Determinant does not span $m + n$ degree forms, so it is not invertible and thus, the determinant is 0. We will now assume that Sylvester's Determinant is 0 and show that A and B must share a root. We know that some non-trivial linear combination of the rows of the determinant are 0.

$$a_1 U^{m-1} A + a_2 U^{m-2} V A + ... + a_m V^{m-1} A - b_1 U^{n-1} B - ... - b_n V^{-1} n B = 0$$

We can now do some factoring,

$$A(a_1 U^{m-1} + a_2 U^{m-2} V + ... + a_m V^{m-1}) - B(b_1 U^{n-1} + ... + b_n V^{n)-1} = 0$$

Notice that $(a_1 U^{m-1} + a_2 U^{m-2} V + ... + a_m V^{m-1})$ is just a form of degree $m-1$ and $(b_1 U^{n-1} + ... + b_n V^{n-1})$ is a form of degree $n-1$. We now have $A\pi = B\tau$ where π is a form of degree $m-1$ and τ is a form of degree $n-1$. Our forms are in $k[U, V]$, so we have unique factorization. Since $deg\pi < degB$ there is at least one root of B that is not a root of π or has a higher multiplicity in B than π. Since $A\pi = B\tau$, it must also be a root of A, thus A and B share a root.