February 17th: The Intersection of Conics and a Pencil of Conics

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Today's lecture covers sections 1.12 to 1.14 in Miles Reid's book *Undergraduate Algebraic Geometry*. These sections focus on the intersection of conics as well as a pencil of conics. In these notes, we will investigate these topics through definitions, propositions, and examples.

Section 1.12: Intersection of Conics

Given 4 points $P_1,\ldots,P_4$ in $\mathbb{P}2$, under the condition that $S2(P_1 \ldots P_4)$ is a 2-dimensional vector space. Recall that S2 is the space of all conics of quadratic form on $\mathbb{R}^3$, which is essentially the set of 3x3 symmetric matrices. Then, by choosing a basis Q_1, Q_2 for $S2(P_1 \ldots P_4)$, we are given two conics C_1 and C_2 such that $C_1 \cap C_2 = \{P_1 \ldots P_4\}$.

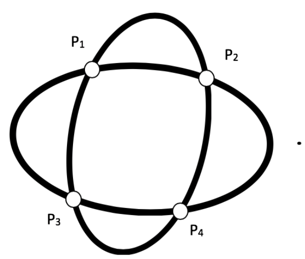
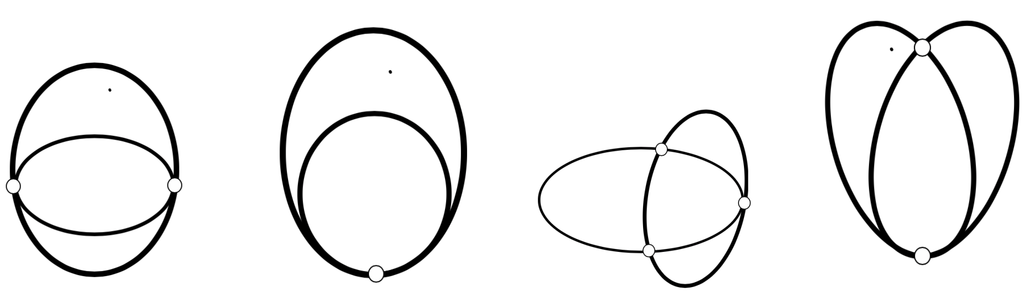


Figure 1: 4 Points of Intersection

Reid's Examples:



Section 1.13: A Pencil of Conics

In section 1.13, the main focus is degenerate conics in a pencil. Let's begin by defining a pencil of conics.

Definition: A Pencil of Conics

A family of the form $\mathrm{C}(\lambda, \mu) = (\lambda Q_1 + \mu Q_2 = 0)$. Each element is a plane curve and the elements are parameterized by $\mathbb{P}^1$. We can think of the ratio $(\lambda : \mu)$ as a point in $\mathbb{P}^1$.

As one might expect, for special values of λ and μ the conic $\mathrm{C}(\lambda, \mu)$ is degenerate. Let's consider detQ for the determinant of the symmetric 3x3 matrix corresponding to the quadratic form Q. When detQ = 0, the conic is degenerate. Then, it is clear that:

$$\mathrm{C}(\lambda, \mu) \text{ is degenerate } \iff \det(\lambda Q_1 + \mu Q_2) = 0.$$

The elements Q1 and Q2 can also be written as 3x3 symmetric matrices. Below this can be expressed as:

$$\mathrm{F}(\lambda, \mu) = det|\lambda \begin{bmatrix} a & b & d \\ b & c & e \\ d & e & f \end{bmatrix} + \mu \begin{bmatrix} a' & b' & d' \\ b' & c' & e' \\ d' & e' & f' \end{bmatrix} | = 0$$

Recall that we write $Q_1 = aX^2 + 2bXY + \ldots + fZ^2$ and Q_2 is of similar form, but uses coefficients $a', b', ..., f'$; note that these are the entries to each of the matrices. In addition, $\mathrm{F}(\lambda, \mu)$ is a homogeneous degree 3 form in λ and μ. By applying what we learned in Section 1.8 to F, we can derive the following proposition:

Proposition:

Suppose $C(\lambda, \mu)$ is a pencil of conics in $\mathbb{P}^2(K)$, with at least one non-degenerate conic. Then the pencil has at most 3 degenerate conics. If $K = \mathbb{R}$, then the pencil has at least one degenerate conic.

Proof:

A cubic form has at least 3 roots by Section 1.8. In addition, over R, it must have at least one root.

Example 1:

Suppose that we start from the pencil of conics generated by the circle, $Q_1 : X^2+Y^2-Z^2 = 0$, and the hyperbola, $Q_2 : X^2 - Y^2 + Z^2 = 0$. Then, we can derive the following: $(\lambda+\mu)X^2 + (\lambda - \mu)Y^2 + (\mu - \lambda)Z^2 = 0$. Consider when $\lambda = 2$ and $\mu = 1$ so we have $3X^2 + Y^2 - Z^2 = 0$.

This can be rewritten as $3X^2 + Y^2 = 1$. No consider when $\lambda = 1$ and $\mu = 2$ so we derive the equation $3X^2 - Y^2 + Z^2 = 0$. Notice that when $\lambda = \mu$, we are given the y axis so $X^2 = 0$.

Now let's compute $F(\lambda, \mu)$. Given the equations for Q_1 and Q_2 above, $F(\lambda, \mu)$ can be written as the following:

$$\text{F}(\lambda, \mu) = det|\lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} + \mu \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} | = det| \begin{bmatrix} \lambda + \mu & 0 & 0 \\ 0 & \lambda - \mu & 0 \\ 0 & 0 & \mu - \lambda \end{bmatrix} |$$

$$= (\lambda + \mu)(\lambda - \mu)(\mu - \lambda)$$

$$= -(\lambda - \mu)^2(\lambda + \mu)$$

Consider when $\lambda = 1$ and $\mu = -1$. Then, we have $(\lambda - \mu)^2(\lambda + \mu) = (1-(-1))^2(1+(-1)) = 0$.

Finally, let's investigate the procedure for finding the points of intersection. First, consider starting from the pencil of conics generated by Q_1, Q_2 in affine form such that $Q_1 = Y^2 + rY + sX + t$ and $Q_2 = Y - X^2$. We will try to find the points $P_1 \dots P_4$ of intersection. Let's plug in $Y = X^2$ into Q_1. Then, Q_1 can be rewritten as $X^4 + rX^2 + sX + t$. This equation is referred to as a "Depressed Quartic." This shows that we can convert every generic quartic into a depressed quartic following a change of variable; this allows us to recover the roots of the original quartic more easily using the depressed quartic.

In order to find the intersection points we must (1) find the 3 ratios $(\lambda : \mu)$ for which $C(\lambda : \mu)$ are degenerate conics, (2) Separate out 2 of the degenerate conics into pairs of lines and (3) the four points P_i are the points of intersection of the lines. When separating out the 2 of the degenerate conics into pairs of line, we get three values of μ/λ for which the conic $\lambda Q_1 + \mu Q_2$ breaks up as line pairs. The cubic equation whose roots are these 3 values is called the "auxiliary cubic" associated with the quartic.