MA357, Spring 2020 — Problem Set 3

This assignment is due on **Friday, February 28**. Primes and factorizations dominate, but there are a few problems on congruences too.

1. NTG, Exercise 2.1.36.

2. The *least common multiple* of two integers a and b is the smallest positive number divisible by both a and b. The usual notation is $\operatorname{lcm}(a, b)$. Prove that

$$\gcd(a, b)\ \operatorname{lcm}(a, b) = ab.$$

(This is easy using unique factorization, a little bit harder without it.)

3. Suppose $n, m \in \mathbb{N}$ and write out their prime factorizations:

$$n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$$
$$m = p_1^{b_1} p_2^{b_2} \cdots p_k^{b_k}$$

with $a_i \geqslant 0$, $b_i \geqslant 0$ (we allow exponent zero in order to be able to use the same list of primes for both numbers). Find the prime factorizations of $\gcd(n, m)$ and $\operatorname{lcm}(m, n)$.

4. Show that if $q = 2^n - 1$ is prime, then n must be prime. Find examples to show that when n is prime then $2^n - 1$ may or may not be prime. (Hint: think factorizations.)

5. Suppose a is an integer, $a \geqslant 2$, $n > 0$, and $a^n + 1$ is prime. Show that n is a power of 2. (Same hint!)

6. Let p be a prime number. Suppose $q = 2^p - 1$ is prime, and let $n = 2^{p-1}q$.

a. Find all the positive proper divisors of n. (A divisor d of n is proper if $d \neq n$.)

b. Show that the sum of the proper divisors of n is equal to n.

c. Find the first three n of this form.

Numbers n such that the sum of the proper divisors of n is equal to n are known as *perfect numbers*. The numbers you will find are the three smallest perfect numbers.

7. For any positive integers n, let $\sigma_0(n)$ be equal to the number of positive divisors of n. Show that if $\gcd(n, m) = 1$ then $\sigma_0(mn) = \sigma_0(m)\sigma_0(n)$. (Functions with this property are called *multiplicative*.)

8. NTG, Exercise 3.5.3. (This is a good example of "it worked once, maybe the same idea will work again.")

9. NTG, Exercises 3.5.10 and 3.5.11.

10. Show that no square has last digit 2, 3, 7, or 8.

11. Suppose m and n are relatively prime, i.e., $\gcd(m, n) = 1$. Show that to say $a \equiv b \pmod{mn}$ is equivalent to the pair of congruences $a \equiv b \pmod{m}$, $a \equiv b \pmod{n}$.

12. NTG, Exercise 4.7.10.

13. Suppose we have $a^2 + b^2 = c^2$ with $\gcd(a, b, c) = 1$. In the previous problem you showed that one of a and b must be even (and the other odd). Suppose b is even.

a. Show that $\gcd(a, b) = 1$ and likewise for the other pairs. In particular, a and c are odd.

b. Rewrite the equation as $b^2 = c^2 - a^3 = (c + a)(c - a)$. What is $\gcd(c + a, c - a)$?

c. Use Problem 1 to conclude that there exist u, v such that $c + a = 2u^2$ and $c - a = 2v^2$.

d. Solve for b in terms of u and v.

e. Find all integer solutions $a^2 + b^2 = c^2$ such that $\gcd(a, b, c) = 1$.

To Explore: We know that the series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \ldots$$

diverges. What happens if we take only the reciprocals of the primes? Does the series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{p} + \ldots$$

converge?

If you can settle that one, what if we select subsets of the primes to work with? For example, how about summing over all p such that $p + 2$ is also prime? Does that series converge?

To Explore: Does the equation $3x^3 + 4y^3 + 5z^3 = 0$ have any solutions in integers? And why is the question interesting?