MA357, Spring 2020 — Problem Set 2

This assignment is due on **Friday, February 21**. Most of the problems are about the gcd and its properties.

1. NTG, Exercise 2.11.18.

2. Find the gcd of 771769 and 32378, and express it as a linear combination of these two numbers.

3. Suppose you know that $\gcd(a, b) = 1$. What can you say about each of the following?

a. $\gcd(a + b, a - b)$

b. $\gcd(a, a + b)$

c. $\gcd(2a, 2b)$

d. $\gcd(2a, b)$

4. Let $n \geqslant 1$ be an integer. Prove that $n! + 1$ and $(n + 1)! + 1$ are always relatively prime.

5. In 1509, DeBouvelles claimed that for every $n \geqslant 1$ at least one of the numbers $6n - 1$ and $6n + 1$ was prime. Find a counterexample to show that he was wrong, then show that there are infinitely many counterexamples (i.e., show that there are infinitely many n such that both $6n - 1$ and $6n + 1$ are composite).

6. One egg timer can time an interval of exactly 5 minutes, and a second can time an interval of exactly 11 minutes. How can we boil an egg for exactly 3 minutes (without buying another timer)?

7. NTG, Exercise 2.11.23.

8. Find all the integer solutions of $15x + 7y = 310$, and then decide how many of them are *positive* integer solutions.

9. NTG, Exercise 2.11.24.

10. Let

$$S = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}.$$

Prove that S is not an integer. (Hint: probably the easiest way is to show that the denominator is divisible by 2.)

11. Are there any prime numbers p of the form $p = n^3 - 1$? If so, how many of them are there?

12. Show that the only n such that n, $n+2$, and $n+4$ are all primes is $n = 3$.

To Explore: Suppose m and n are positive integers and we use the Euclidean algorithm to compute their gcd. If m and n are big, we should try to estimate how many steps the algorithm is going to take before we launch into it. Try to find a way to estimate the maximum number of steps that will be necessary.

Hint: The worst case scenario is when we get $q = 1$ in all the divisions. If that happens, what can you say and m and n?

To Explore: Let a_1, a_2, ..., a_n be positive integers. Study the theory of diophantine equations of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b.$$

If we restrict the solutions to *positive* integers, i.e., we assume $x_i \geqslant 0$, this is sometimes known as the postage stamp problem: think of a_1, a_2, etc. as the values of the stamps you have, and of b as the amount of postage you want to put onto an envelope.

To Explore: Bertrand conjectured, and Chebychev proved, that for any integer $n \geqslant 2$ there is always a prime number p such that $n < p < 2n$. Figure out how to prove this.