

Positive-homogeneous operators, heat kernel estimates and the Legendre-Fenchel transform

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Abstract

We consider a class of homogeneous partial differential operators on a finite-dimensional vector space and study their associated heat kernels. The heat kernels for this general class of operators are seen to arise naturally as the limiting objects of the convolution powers of complex-valued functions on the square lattice in the way that the classical heat kernel arises in the (local) central limit theorem. These so-called positive-homogeneous operators generalize the class of semi-elliptic operators in the sense that the definition is coordinate-free. More generally, we introduce a class of variable-coefficient operators, each of which is uniformly comparable to a positive-homogeneous operator, and we study the corresponding Cauchy problem for the heat equation. Under the assumption that such an operator has Hölder continuous coefficients, we construct a fundamental solution to its heat equation by the method of E. E. Levi, adapted to parabolic systems by A. Friedman and S. D. Eidelman. Though our results in this direction are implied by the long-known results of S. D. Eidelman for $2\vec{b}$ -parabolic systems, our focus is to highlight the role played by the Legendre-Fenchel transform in heat kernel estimates. Specifically, we show that the fundamental solution satisfies an off-diagonal estimate, i.e., a heat kernel estimate, written in terms of the Legendre-Fenchel transform of the operator's principal symbol—an estimate which is seen to be sharp in many cases.

Keywords: Semi-elliptic operators, quasi-elliptic operators, $2\vec{b}$ -parabolic operators, heat kernel estimates, Legendre-Fenchel transform.

Mathematics Subject Classification: Primary 35H30; Secondary 35K25.

1 Introduction

In this article, we consider a class of homogeneous partial differential operators on a finite dimensional vector space and study their associated heat kernels. These operators, which we call nondegenerate-homogeneous operators, are seen to generalize the well-studied classes of semi-elliptic operators introduced by F. Browder [13], also known as quasielliptic operators [62], and a special “positive” subclass of semi-elliptic operators which appear as the spatial part of S. D. Eidelman's $2\vec{b}$ -parabolic operators [36]. In particular, this class of operators contains all integer powers of the Laplacian. We begin this introduction by motivating the study of these homogeneous operators by first demonstrating the natural appearance of their heat kernels in the study of convolution powers of complex valued functions. To this end, consider a

finitely supported function $\phi : \mathbb{Z}^d \rightarrow \mathbb{C}$ and define its convolution powers iteratively by

$$\phi^{(n)}(x) = \sum_{y \in \mathbb{Z}^d} \phi^{(n-1)}(x-y)\phi(y)$$

for $x \in \mathbb{Z}^d$ where $\phi^{(1)} = \phi$. In the special case that ϕ is a probability distribution, i.e., ϕ is non-negative and has unit mass, ϕ drives a random walk on \mathbb{Z}^d whose n th-step transition kernels are given by $k_n(x, y) = \phi^{(n)}(y-x)$. Under certain mild conditions on the random walk, $\phi^{(n)}$ is well-approximated by a single Gaussian density; this is the classical local limit theorem. Specifically, for a symmetric, aperiodic and irreducible random walk, the theorem states that

$$\phi^{(n)}(x) = n^{-d/2}G_\phi(x/\sqrt{n}) + o(n^{-d/2})$$

uniformly for $x \in \mathbb{Z}^d$, where G_ϕ is the generalized Gaussian density

$$G_\phi(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \exp(-\xi \cdot C_\phi \xi) e^{-ix \cdot \xi} d\xi = \frac{1}{(2\pi)^{d/2} \sqrt{\det C_\phi}} \exp\left(-\frac{x \cdot C_\phi^{-1} x}{2}\right); \quad (1)$$

here, C_ϕ is the positive definite covariance matrix associated to ϕ and \cdot denotes the dot product [48, 53, 57]. The canonical example is that in which $C_\phi = I$ (e.g. Simple Random Walk) and in this case $\phi^{(n)}$ is approximated by the so-called heat kernel

$$K_{(-\Delta)}^n(x) = n^{-d/2}G_\phi(x/\sqrt{n}) = (2\pi n)^{-d/2} \exp\left(-\frac{|x|^2}{2n}\right).$$

In addition to its natural appearance as the *attractor* in the local limit theorem, $K_{(-\Delta)}^t(x)$ is a fundamental solution to the heat equation

$$\partial_t + (-\Delta) = 0.$$

In fact, this connection to random walk underlies the heat equation's probabilistic/diffusive interpretation. Beyond the probabilistic setting, this link between convolution powers and fundamental solutions to partial differential equations persists as can be seen in the following examples.

Example 1. Consider $\phi : \mathbb{Z}^2 \rightarrow \mathbb{C}$ defined by

$$\phi(x_1, x_2) = \frac{1}{22 + 2\sqrt{3}} \times \begin{cases} 8 & (x_1, x_2) = (0, 0) \\ 5 + \sqrt{3} & (x_1, x_2) = (\pm 1, 0) \\ -2 & (x_1, x_2) = (\pm 2, 0) \\ i(\sqrt{3} - 1) & (x_1, x_2) = (\pm 1, -1) \\ -i(\sqrt{3} - 1) & (x_1, x_2) = (\pm 1, 1) \\ 2 \mp 2i & (x_1, x_2) = (0, \pm 1) \\ 0 & \text{otherwise.} \end{cases}$$

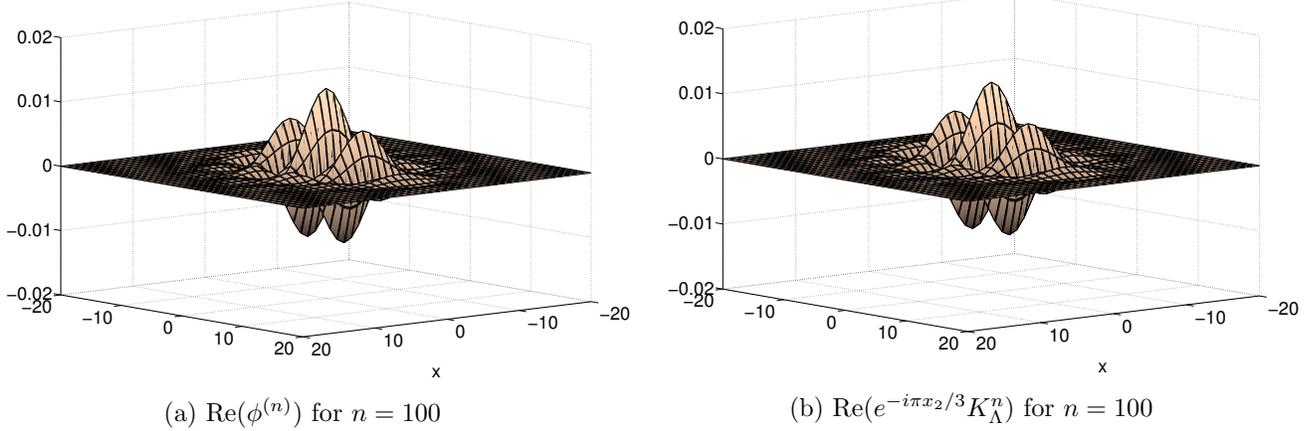


Figure 1: The graphs of $\text{Re}(\phi^{(n)})$ and $\text{Re}(e^{-i\pi x_2/3}K_\Lambda^n)$ for $n = 100$.

Analogous to the probabilistic setting, the large n behavior of $\phi^{(n)}$ is described by a generalized local limit theorem in which the attractor is a fundamental solution to a heat-type equation. Specifically, the following local limit theorem holds (see [53] for details):

$$\phi^{(n)}(x_1, x_2) = e^{-i\pi x_2/3}K_\Lambda^n(x_1, x_2) + o(n^{-3/4})$$

uniformly for $(x_1, x_2) \in \mathbb{Z}^2$ where K_Λ is the “heat” kernel for the heat-type equation $\partial_t + \Lambda = 0$ where

$$\Lambda = \frac{1}{22 + 2\sqrt{3}} \left(2\partial_{x_1}^4 - i(\sqrt{3} - 1)\partial_{x_1}^2\partial_{x_2} - 4\partial_{x_2}^2 \right).$$

This local limit theorem is illustrated in Figure 1 which shows $\text{Re}(\phi^{(n)})$ and the approximation $\text{Re}(e^{-i\pi x_2/3}K_\Lambda^n)$ when $n = 100$.

Example 2. Consider $\phi : \mathbb{Z}^2 \rightarrow \mathbb{R}$ defined by $\phi = (\phi_1 + \phi_2)/512$, where

$$\phi_1(x_1, x_2) = \begin{cases} 326 & (x_1, x_2) = (0, 0) \\ 20 & (x_1, x_2) = (\pm 2, 0) \\ 1 & (x_1, x_2) = (\pm 4, 0) \\ 64 & (x_1, x_2) = (0, \pm 1) \\ -16 & (x_1, x_2) = (0, \pm 2) \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \phi_2(x_1, x_2) = \begin{cases} 76 & (x_1, x_2) = (1, 0) \\ 52 & (x_1, x_2) = (-1, 0) \\ \mp 4 & (x_1, x_2) = (\pm 3, 0) \\ \mp 6 & (x_1, x_2) = (\pm 1, 1) \\ \mp 6 & (x_1, x_2) = (\pm 1, -1) \\ \pm 2 & (x_1, x_2) = (\pm 3, 1) \\ \pm 2 & (x_1, x_2) = (\pm 3, -1) \\ 0 & \text{otherwise.} \end{cases}$$

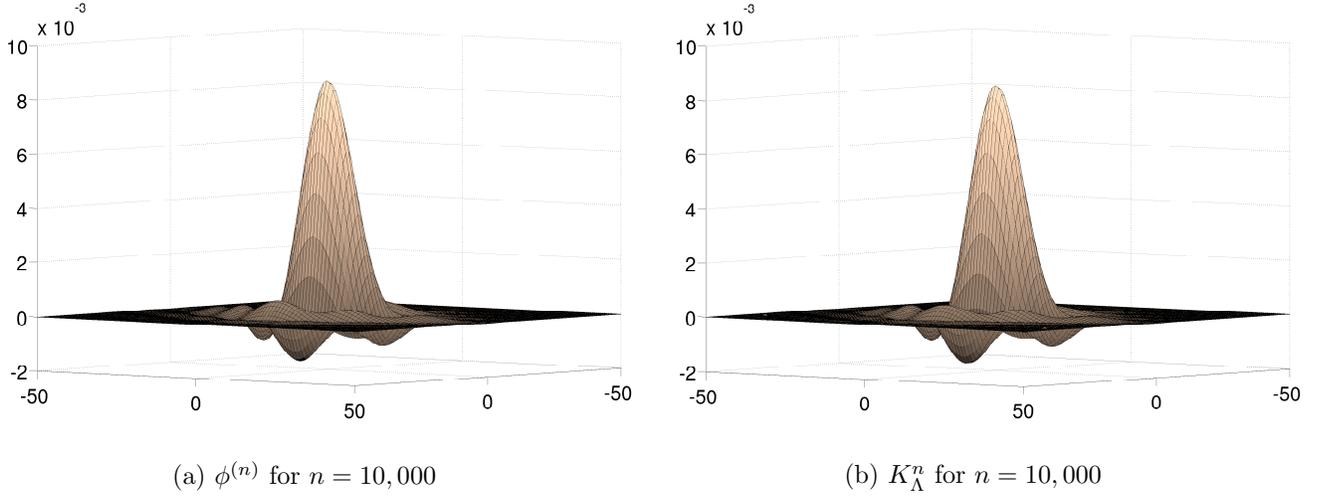


Figure 2: The graphs of $\phi^{(n)}$ and K_{Λ}^n for $n = 10,000$.

In this example, the following local limit theorem, which is illustrated by Figure 2, describes the limiting behavior of $\phi^{(n)}$. We have

$$\phi^{(n)}(x_1, x_2) = K_{\Lambda}^n(x_1, x_2) + o(n^{-5/12})$$

uniformly for $(x_1, x_2) \in \mathbb{Z}^2$ where K_{Λ} is again a fundamental solution to $\partial_t + \Lambda = 0$ where, in this case,

$$\Lambda = \frac{1}{64} (-\partial_{x_1}^6 + 2\partial_{x_2}^4 + 2\partial_{x_1}^3 \partial_{x_2}^2).$$

Example 3. Consider $\phi : \mathbb{Z}^2 \rightarrow \mathbb{R}$ defined by

$$\phi(x, y) = \begin{cases} 3/8 & (x_1, x_2) = (0, 0) \\ 1/8 & (x_1, x_2) = \pm(1, 1) \\ 1/4 & (x_1, x_2) = \pm(1, -1) \\ -1/16 & (x_1, x_2) = \pm(2, -2) \\ 0 & \text{otherwise.} \end{cases}$$

Here, the following local limit theorem is valid:

$$\phi^{(n)}(x_1, x_2) = \left(1 + e^{i\pi(x_1+x_2)}\right) K_{\Lambda}^n(x_1, x_2) + o(n^{-3/4})$$

uniformly for $(x_1, x_2) \in \mathbb{Z}^2$. Here again, the attractor K_{Λ} is the fundamental solution to $\partial_t + \Lambda = 0$ where

$$\Lambda = -\frac{1}{8}\partial_{x_1}^2 + \frac{23}{384}\partial_{x_1}^4 - \frac{1}{4}\partial_{x_1}\partial_{x_2} - \frac{25}{96}\partial_{x_1}^3\partial_{x_2} - \frac{1}{8}\partial_{x_2}^2 + \frac{23}{64}\partial_{x_1}^2\partial_{x_2}^2 - \frac{25}{96}\partial_{x_1}\partial_{x_2}^3 + \frac{23}{384}\partial_{x_2}^4.$$

The operators appearing in the above examples share two important properties: homogeneity and positivity. While we make these notions precise in the next section, loosely speaking, homogeneity is the property that Λ “plays well” with some dilation structure on \mathbb{R}^d , though this structure is different in each example. Further, homogeneity for Λ is reflected by an analogous one for the corresponding heat kernel K_Λ ; in fact, the specific dilation structure is, in some sense, selected by $\phi^{(n)}$ as $n \rightarrow \infty$ and leads to the corresponding local limit theorem. We encourage the reader to see the recent article [53] for a more thorough study of these examples and, in general, a more thorough study of local limit theorems. As we have often found—through local limit theorems and otherwise—knowledge of the attractor K_Λ informs our study of convolution powers (see Theorem 1.6 and Section 5.1 of [53]).

The prototypical examples of homogeneous operators considered in this article are the so-called semi-elliptic operators originally introduced by F. Browder in [13] and shortly appearing thereafter in L. Hörmander’s treatise on linear partial differential operators [45, 46]. Given d -tuple of positive integers $\mathbf{n} = (n_1, n_2, \dots, n_d) \in \mathbb{N}_+^d$ and a multi-index $\beta = (\beta_1, \beta_2, \dots, \beta_d) \in \mathbb{N}^d$, set $|\beta : \mathbf{n}| = \sum_{k=1}^d \beta_k/n_k$. Consider the constant coefficient partial differential operator

$$\Lambda = \sum_{|\beta : \mathbf{n}| \leq 1} a_\beta D^\beta$$

with principal part (relative to \mathbf{n})

$$\Lambda_p = \sum_{|\beta : \mathbf{n}| = 1} a_\beta D^\beta,$$

where $a_\beta \in \mathbb{C}$ and $D^\beta = (i\partial_{x_1})^{\beta_1} (i\partial_{x_2})^{\beta_2} \dots (i\partial_{x_d})^{\beta_d}$ for each multi-index $\beta \in \mathbb{N}^d$. Such an operator is said to be *semi-elliptic* if the symbol of Λ_p , defined by $P_p(\xi) = \sum_{|\beta : \mathbf{n}| = 1} a_\beta \xi^\beta$ for $\xi \in \mathbb{R}^d$, is non-vanishing away from the origin. If Λ satisfies the stronger condition that $\operatorname{Re} P_p(\xi)$ is strictly positive away from the origin, we say that it is *positive-semi-elliptic*. What seems to be the most important property of semi-elliptic operators is that their principal part Λ_p is homogeneous in the following sense: If given any smooth function f we put $\delta_t(f)(x) = f(t^{1/n_1}x_1, t^{1/n_2}x_2, \dots, t^{1/n_d}x_d)$ for all $t > 0$ and $x = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$, then

$$t\Lambda = \delta_{1/t} \circ \Lambda_p \circ \delta_t$$

for all $t > 0$. This homogeneous structure was used explicitly in the work of F. Browder and L. Hörmander and, in this article, we generalize this notion. Our generalization captures the operators appearing in Examples 1, 2 and 3.

As mentioned above, the class of semi-elliptic operators was introduced by F. Browder in [13] who studied spectral asymptotics for a related class of variable-coefficient operators (operators of constant strength). Semi-elliptic operators appeared later in L. Hörmander’s text [45] as model examples of hypoelliptic operators on \mathbb{R}^d beyond the class of elliptic operators. Around the same time L. R. Volevich [62] independently introduced the same class of operators but instead called them “quasi-elliptic”. Since then, the theory of semi-elliptic operators, and hence quasi-elliptic operators, has reached a high level of sophistication and we refer the reader to the articles [1–5, 13, 43–47, 58, 60], which use the term semi-elliptic, and the articles [10–12, 15, 18–33, 40, 50, 52, 59, 61, 62], which use the term quasi-elliptic, for an account of this theory. We would also like to point to the 1971 paper of M. Troisi [59] which gives a more complete list of

references (pertaining to quasi-elliptic operators).

Shortly after F. Browder's paper [13] appeared, S. D. Eidelman considered a subclass of semi-elliptic operators on $\mathbb{R}^{d+1} = \mathbb{R} \oplus \mathbb{R}^d$ (and systems thereof) of the form

$$\partial_t + \sum_{|\beta:2\mathbf{m}|\leq 1} a_\beta D^\beta = \partial_t + \sum_{|\beta:\mathbf{m}|\leq 2} a_\beta D^\beta, \quad (2)$$

where $\mathbf{m} \in \mathbb{N}_+^d$ and the coefficients a_β are functions of x and t . Such an operator is said to be $2\mathbf{m}$ -parabolic if its spatial part, $\sum_{|\beta:2\mathbf{m}|\leq 1} a_\beta D^\beta$, is (uniformly) positive-semi-elliptic. We note however that Eidelman's work and the existing literature refer exclusively to $2\vec{b}$ -parabolic operators, i.e., where $\mathbf{m} = \vec{b}$, and for consistency we write $2\vec{b}$ -parabolic henceforth [36, 37]. The relationship between positive-semi-elliptic operators and $2\vec{b}$ -parabolic operators is analogous to the relationship between the Laplacian and the heat operator and, in the context of this article, the relationship between nondegenerate-homogeneous and positive-homogeneous operators described by Proposition 2.4. The theory of $2\vec{b}$ -parabolic operators, which generalizes the theory of parabolic partial differential equations (and systems), has seen significant advancement by a number of mathematicians since Eidelman's original work. We encourage the reader to see the recent text [37] which provides an account of this theory and an exhaustive list of references. It should be noted however that the literature encompassing semi-elliptic operators and quasi-elliptic operators, as far as we can tell, has very few cross-references to the literature on $2\vec{b}$ -parabolic operators beyond the 1960's. We suspect that the absence of cross-references is due to the distinctness of vocabulary.

Returning to our discussion of convolution power examples, we note that the operators appearing in Examples 1 and 2 are both positive-semi-elliptic and consist only of their principal parts. This is easily verified, for $\mathbf{n} = (4, 2) = 2(2, 1)$ in Example 1 and $\mathbf{n} = (6, 4) = 2(3, 2)$ in Example 2. In contrast to Examples 1 and 2, the operator Λ which appears in Example 3 is not semi-elliptic in the given coordinate system. After careful study, the Λ appearing in Example 3 can be written equivalently as

$$\Lambda = -\frac{1}{8}\partial_{v_1}^2 + \frac{23}{384}\partial_{v_2}^4 \quad (3)$$

where ∂_{v_1} is the directional derivative in the $v_1 = (1, 1)$ direction and ∂_{v_2} is the directional derivative in the $v_2 = (1, -1)$ direction. In this way, Λ is seen to be semi-elliptic with respect to some basis $\{v_1, v_2\}$ of \mathbb{R}^2 . For this reason, our formulation of nondegenerate-homogeneous operators (and positive-homogeneous operators), given in the next section, is made in a basis independent way.

The subject of this paper is an account of positive-homogeneous operators, a class of operators which generalize semi-elliptic operators, and their corresponding heat equations. In Section 2, we introduce positive-homogeneous operators and study their basic properties; therein, we show that each positive-homogeneous operator is semi-elliptic in some coordinate system. Section 3 develops the necessary background to introduce the class of variable-coefficient operators studied in this article; this is the class of $(2\mathbf{m}, \mathbf{v})$ -positive-semi-elliptic operators introduced in Section 4—each of which is comparable to a constant-coefficient positive-homogeneous operator. In Section 5, we study the heat equations corresponding to uniformly $(2\mathbf{m}, \mathbf{v})$ -positive-semi-elliptic operators with Hölder continuous coefficients. Specifically, we use

the famous method of E. E. Levi, adapted to parabolic systems by A. Friedman and S. D. Eidelman, to construct a fundamental solution to the corresponding heat equation. Our results in this direction are captured by those of S. D. Eidelman [36] and the works of his collaborators, notably S. D. Ivashyshen and A. N. Kochubei [37], concerning $2\vec{b}$ -parabolic systems. Our focus in this presentation is to highlight the essential role played by the Legendre-Fenchel transform in heat kernel estimates which, to our knowledge, has not been pointed out in the context of semi-elliptic operators. In a forthcoming work, we study an analogous class of operators, written in divergence form, with measurable-coefficients and their corresponding heat kernels. This class of measurable-coefficient operators does not appear to have been previously studied. The results presented here, using the Legendre-Fenchel transform, provides the background and context for our work there.

1.1 Preliminaries

Fourier Analysis: Our setting is a real d -dimensional vector space \mathbb{V} equipped with Haar (Lebesgue) measure dx and the standard smooth structure; we do not affix \mathbb{V} with a norm or basis. The dual space of \mathbb{V} is denoted by \mathbb{V}^* and the dual pairing is denoted by $\xi(x)$ for $x \in \mathbb{V}$ and $\xi \in \mathbb{V}^*$. Let $d\xi$ be the Haar measure on \mathbb{V}^* which we take to be normalized so that our convention for the Fourier transform and inverse Fourier transform, given below, makes each unitary. Throughout this article, all functions on \mathbb{V} and \mathbb{V}^* are understood to be complex-valued. The usual Lebesgue spaces are denoted by $L^p(\mathbb{V}) = L^p(\mathbb{V}, dx)$ and equipped with their usual norms $\|\cdot\|_p$ for $1 \leq p \leq \infty$. In the case that $p = 2$, the corresponding inner product on $L^2(\mathbb{V})$ is denoted by $\langle \cdot, \cdot \rangle$. Of course, we will also work with $L^2(\mathbb{V}^*) := L^2(\mathbb{V}^*, d\xi)$; here the L^2 -norm and inner product will be denoted by $\|\cdot\|_{2^*}$ and $\langle \cdot, \cdot \rangle_*$ respectively. The Fourier transform $\mathcal{F} : L^2(\mathbb{V}) \rightarrow L^2(\mathbb{V}^*)$ and inverse Fourier transform $\mathcal{F}^{-1} : L^2(\mathbb{V}^*) \rightarrow L^2(\mathbb{V})$ are initially defined for Schwartz functions $f \in \mathcal{S}(\mathbb{V})$ and $g \in \mathcal{S}(\mathbb{V}^*)$ by

$$\mathcal{F}(f)(\xi) = \hat{f}(\xi) = \int_{\mathbb{V}} e^{i\xi(x)} f(x) dx \quad \text{and} \quad \mathcal{F}^{-1}(g)(x) = \check{g}(x) = \int_{\mathbb{V}^*} e^{-i\xi(x)} g(\xi) d\xi$$

for $\xi \in \mathbb{V}^*$ and $x \in \mathbb{V}$ respectively.

For the remainder of this article (mainly when duality isn't of interest), W stands for any real d -dimensional vector space (and so is interchangeable with \mathbb{V} or \mathbb{V}^*). For a non-empty open set $\Omega \subseteq W$, we denote by $C(\Omega)$ and $C_b(\Omega)$ the set of continuous functions on Ω and bounded continuous functions on Ω , respectively. The set of smooth functions on Ω is denoted by $C^\infty(\Omega)$ and the set of compactly supported smooth functions on Ω is denoted by $C_0^\infty(\Omega)$. We denote by $\mathcal{D}'(\Omega)$ the space of distributions on Ω ; this is dual to the space $C_0^\infty(\Omega)$ equipped with its usual topology given by seminorms. A partial differential operator H on W is said to be *hypoelliptic* if it satisfies the following property: Given any open set $\Omega \subseteq W$ and any distribution $u \in \mathcal{D}'(\Omega)$ which satisfies $Hu = 0$ in Ω , then necessarily $u \in C^\infty(\Omega)$.

Dilation Structure: Denote by $\text{End}(W)$ and $\text{Gl}(W)$ the set of endomorphisms and isomorphisms of W respectively. Given $E \in \text{End}(W)$, we consider the one-parameter group $\{t^E\}_{t>0} \subseteq \text{Gl}(W)$ defined by

$$t^E = \exp((\log t)E) = \sum_{k=0}^{\infty} \frac{(\log t)^k}{k!} E^k$$

for $t > 0$. These one-parameter subgroups of $\text{Gl}(W)$ allow us to define continuous one-parameter groups of operators on the space of distributions as follows: Given $E \in \text{End}(W)$ and $t > 0$, first define $\delta_t^E(f)$ for $f \in C_0^\infty(W)$ by $\delta_t^E(f)(x) = f(t^E x)$ for $x \in W$. Extending this to the space of distribution on W in the usual way, the collection $\{\delta_t^E\}_{t>0}$ is a continuous one-parameter group of operators on $\mathcal{D}'(W)$; it will allow us to define homogeneity for partial differential operators in the next section.

Linear Algebra and Polynomials: Given a basis $\mathbf{w} = \{w_1, w_2, \dots, w_d\}$ of W , we define the map $\phi_{\mathbf{w}} : W \rightarrow \mathbb{R}^d$ by setting $\phi_{\mathbf{w}}(w) = (x_1, x_2, \dots, x_d)$ whenever $w = \sum_{l=1}^d x_l w_l$. This map defines a global coordinate system on W ; any such coordinate system is said to be a linear coordinate system on W . By definition, a polynomial on W is a function $P : W \rightarrow \mathbb{C}$ that is a polynomial function in every (and hence any) linear coordinate system on W . A polynomial P on W is called a nondegenerate polynomial if $P(w) \neq 0$ for all $w \neq 0$. Further, P is called a positive-definite polynomial if its real part, $R = \text{Re } P$, is non-negative and has $R(w) = 0$ only when $w = 0$.

The Rest: Finally, the symbols $\mathbb{R}, \mathbb{C}, \mathbb{Z}$ mean what they usually do, \mathbb{N} denotes the set of non-negative integers and $\mathbb{I} = [0, 1] \subseteq \mathbb{R}$. The symbols $\mathbb{R}_+, \mathbb{N}_+$ and \mathbb{I}_+ denote the set of strictly positive elements of \mathbb{R}, \mathbb{N} and \mathbb{I} respectively. Likewise, $\mathbb{R}_+^d, \mathbb{N}_+^d$ and \mathbb{I}_+^d respectively denote the set of d -tuples of these aforementioned sets. We say that two real-valued functions f and g on a set X are comparable if, for some positive constant C , $C^{-1}f(x) \leq g(x) \leq Cf(x)$ for all $x \in X$; in this case we write $f \asymp g$. Adopting the summation notation for semi-elliptic operators of L. Hörmander's treatise [46], for a fixed $\mathbf{n} = (n_1, n_2, \dots, n_d) \in \mathbb{N}_+^d$, we write

$$|\beta : \mathbf{n}| = \sum_{k=1}^d \frac{\beta_k}{m_k}.$$

for all multi-indices $\beta = (\beta_1, \beta_2, \dots, \beta_d) \in \mathbb{N}^d$. Finally, throughout the estimates made in this article, constants denoted by C will change from line to line without explicit mention.

2 Homogeneous operators

In this section we introduce two important classes of homogeneous constant-coefficient on \mathbb{V} . These operators will serve as “model” operators in our theory in the way that integer powers of the Laplacian serves a model operators in the elliptic theory of partial differential equations. To this end, let Λ be a constant-coefficient partial differential operator on \mathbb{V} and let $P : \mathbb{V}^* \rightarrow \mathbb{C}$ be its symbol. Specifically, P is the polynomial on \mathbb{V}^* defined by $P(\xi) = e^{-i\xi(x)} \Lambda(e^{i\xi(x)})$ for $\xi \in \mathbb{V}^*$ (this is independent of $x \in \mathbb{V}$ precisely because Λ is a constant-coefficient operator). We first introduce the following notion of homogeneity of operators; it is mirrored by an analogous notion for symbols which we define shortly.

Definition 2.1. *Given $E \in \text{End}(\mathbb{V})$, we say that a constant-coefficient partial differential operator Λ is homogeneous with respect to the one-parameter group $\{\delta_t^E\}$ if*

$$\delta_{1/t}^E \circ \Lambda \circ \delta_t^E = t\Lambda$$

for all $t > 0$; in this case we say that E is a member of the exponent set of Λ and write $E \in \text{Exp}(\Lambda)$.

A constant-coefficient partial differential operator Λ need not be homogeneous with respect to a unique one-parameter group $\{\delta_t^E\}$, i.e., $\text{Exp}(\Lambda)$ is not necessarily a singleton. For instance, it is easily verified that, for the Laplacian $-\Delta$ on \mathbb{R}^d ,

$$\text{Exp}(-\Delta) = 2^{-1}I + \mathfrak{o}_d$$

where I is the identity and \mathfrak{o}_d is the Lie algebra of the orthogonal group, i.e., is given by the set of skew-symmetric matrices. Despite this lack of uniqueness, when Λ is equipped with a nondegenerateness condition (see Definition 2.2), we will find that trace is the same for each member of $\text{Exp}(\Lambda)$ and this allows us to uniquely define an “order” for Λ ; this is Lemma 2.10.

Given a constant coefficient operator Λ with symbol P , one can quickly verify that $E \in \text{Exp}(\Lambda)$ if and only if

$$tP(\xi) = P(t^F\xi) \tag{4}$$

for all $t > 0$ and $\xi \in \mathbb{V}^*$ where $F = E^*$ is the adjoint of E . More generally, if P is any continuous function on W and (4) is satisfied for some $F \in \text{End}(\mathbb{V}^*)$, we say that P is *homogeneous with respect to* $\{t^F\}$ and write $F \in \text{Exp}(P)$. This admitted slight abuse of notation should not cause confusion. In this language, we see that $E \in \text{Exp}(\Lambda)$ if and only if $E^* \in \text{Exp}(P)$.

We remark that the notion of homogeneity defined above is similar to that put forth for homogeneous operators on homogeneous (Lie) groups, e.g., Rockland operators [38]. The difference is mostly a matter of perspective: A homogeneous group G is equipped with a fixed dilation structure, i.e., it comes with a one-parameter group $\{\delta_t\}$, and homogeneity of operators is defined with respect to this fixed dilation structure. By contrast, we fix no dilation structure on \mathbb{V} and formulate homogeneity in terms of an operator Λ and the existence of a one-parameter group $\{\delta_t^E\}$ that “plays” well with Λ in sense defined above. As seen in the study of convolution powers on the square lattice (see [53]), it useful to have this freedom.

Definition 2.2. *Let Λ be constant-coefficient partial differential operator on \mathbb{V} with symbol P . We say that Λ is a nondegenerate-homogeneous operator if P is a nondegenerate polynomial and $\text{Exp}(\Lambda)$ contains a diagonalizable endomorphism. We say that Λ is a positive-homogeneous operator if P is a positive-definite polynomial and $\text{Exp}(\Lambda)$ contains a diagonalizable endomorphism.*

For any polynomial P on a finite-dimensional vector space W , P is said to be *nondegenerate-homogeneous* if P is nondegenerate and $\text{Exp}(P)$, defined as the set of $F \in \text{End}(W)$ for which (4) holds, contains a diagonalizable endomorphism. We say that P is *positive-homogeneous* if it is a positive-definite polynomial and $\text{Exp}(P)$ contains a diagonalizable endomorphism. In this language, we have the following proposition.

Proposition 2.3. *Let Λ be a positive homogeneous operator on \mathbb{V} with symbol P . Then Λ is a nondegenerate-homogeneous operator if and only if P is a nondegenerate-homogeneous polynomial. Further, Λ is a positive-homogeneous operator if and only if P is a positive-homogeneous polynomial.*

Proof. Since the adjectives “nondegenerate” and “positive”, in the sense of both operators and polynomials, are defined in terms of the symbol P , all that needs to be verified is that $\text{Exp}(\Lambda)$ contains a diagonalizable endomorphism if and only if $\text{Exp}(P)$ contains a diagonalizable endomorphism. Upon recalling that $E \in \text{Exp}(\Lambda)$ if and only if $E^* \in \text{Exp}(P)$, this equivalence is verified by simply noting that diagonalizability is preserved under taking adjoints. \square

Remark 1. To capture the class of nondegenerate-homogeneous operators (or positive-homogeneous operators), in addition to requiring that the symbol P of an operator Λ be nondegenerate (or positive-definite), one can instead demand only that $\text{Exp}(\Lambda)$ contains an endomorphism whose characteristic polynomial factors over \mathbb{R} or, equivalently, whose spectrum is real. This a priori weaker condition is seen to be sufficient by an argument which makes use of the Jordan-Chevalley decomposition. In the positive-homogeneous case, this argument is carried out in [53] (specifically Proposition 2.2) wherein positive-homogeneous operators are first defined by this (a priori weaker) condition. For the nondegenerate case, the same argument pushes through with very little modification.

We observe easily that all positive-homogeneous operators are nondegenerate-homogeneous. It is the “heat” kernels corresponding to positive-homogeneous operators that naturally appear in [53] as the attractors of convolution powers of complex-valued functions. The following proposition highlights the interplay between positive-homogeneity and nondegenerate-homogeneity for an operator Λ on \mathbb{V} and its corresponding “heat” operator $\partial_t + \Lambda$ on $\mathbb{R} \oplus \mathbb{V}$.

Proposition 2.4. *Let Λ be a constant-coefficient partial differential operator on \mathbb{V} whose exponent set $\text{Exp}(\Lambda)$ contains a diagonalizable endomorphism. Let P be the symbol of Λ , set $R = \text{Re } P$, and assume that there exists $\xi \in \mathbb{V}^*$ for which $R(\xi) > 0$. We have the following dichotomy: Λ is a positive-homogeneous operator on \mathbb{V} if and only if $\partial_t + \Lambda$ is a nondegenerate-homogeneous operator on $\mathbb{R} \oplus \mathbb{V}$.*

Proof. Given a diagonalizable endomorphism $E \in \text{Exp}(\Lambda)$, set $E_1 = I \oplus E$ where I is the identity on \mathbb{R} . Obviously, E_1 is diagonalizable. Further, for any $f \in C_0^\infty(\mathbb{R} \oplus \mathbb{V})$,

$$\begin{aligned} ((\partial_t + \Lambda) \circ \delta_s^{E_1})(f)(t, x) &= (\partial_t (f(st, s^E x)) + \Lambda (f(st, s^E x))) \\ &= s(\partial_t + \Lambda)(f)(st, s^E x) = s(\delta_s^{E_1} \circ (\partial_t + \Lambda))(f)(t, x) \end{aligned}$$

for all $s > 0$ and $(t, x) \in \mathbb{R} \oplus \mathbb{V}$. Hence

$$\delta_{1/s}^{E_1} \circ (\partial_t + \Lambda) \circ \delta_t^{E_1} = s(\partial_t + \Lambda)$$

for all $s > 0$ and therefore $E_1 \in \text{Exp}(\partial_t + \Lambda)$.

It remains to show that P is positive-definite if and only if the symbol of $\partial_t + \Lambda$ is nondegenerate. To this end, we first compute the symbol of $\partial_t + \Lambda$ which we denote by Q . Since the dual space of $\mathbb{R} \oplus \mathbb{V}$ is isomorphic to $\mathbb{R} \oplus \mathbb{V}^*$, the characters of $\mathbb{R} \oplus \mathbb{V}$ are represented by the collection of maps $(\mathbb{R} \oplus \mathbb{V}) \ni (t, x) \mapsto \exp(-i(\tau t + \xi(x)))$ where $(\tau, \xi) \in \mathbb{R} \oplus \mathbb{V}^*$. Consequently,

$$Q(\tau, \xi) = e^{-i(\tau t + \xi(x))} (\partial_t + \Lambda) (e^{i(\tau t + \xi(x))}) = i\tau + P(\xi)$$

for $(\tau, \xi) \in \mathbb{R} \oplus \mathbb{V}^*$. We note that $P(0) = 0$ because $E^* \in \text{Exp}(P)$; in fact, this happens whenever $\text{Exp}(P)$ is non-empty. Now if P is a positive-definite polynomial, $\text{Re } Q(\tau, \xi) = \text{Re } P(\xi) = R(\xi) > 0$ whenever $\xi \neq 0$. Thus to verify that Q is a nondegenerate polynomial, we simply must verify that $Q(\tau, 0) \neq 0$ for all non-zero $\tau \in \mathbb{R}$. This is easy to see because, in light of the above fact, $Q(\tau, 0) = i\tau + P(0) = i\tau \neq 0$ whenever $\tau \neq 0$ and hence Q is nondegenerate. For the other direction, we demonstrate the validity of the contrapositive statement. Assuming that P is not positive-definite, an application of the intermediate value theorem, using the condition that $R(\xi) > 0$ for some $\xi \in \mathbb{V}^*$, guarantees that $R(\eta) = 0$ for some non-zero $\eta \in \mathbb{V}^*$. Here, we observe that $Q(\tau, \eta) = i(\tau + \text{Im } P(\eta)) = 0$ when $(\tau, \eta) = (-\text{Im } P(\eta), \eta)$ and hence Q is not nondegenerate. \square

We will soon return to the discussion surrounding a positive-homogeneous operator Λ and its heat operator $\partial_t + \Lambda$. It is useful to first provide representation formulas for nondegenerate-homogeneous and positive-homogeneous operators. Such representations connect our homogeneous operators to the class of semi-elliptic operators discussed in the introduction. To this end, we define the “base” operators on \mathbb{V} . First, for any element $u \in \mathbb{V}$, we consider the differential operator $D_u : \mathcal{D}'(\mathbb{V}) \rightarrow \mathcal{D}'(\mathbb{V})$ defined originally for $f \in C_0^\infty(\mathbb{V})$ by

$$(D_u f)(x) = i \frac{\partial f}{\partial u}(x) = i \left(\lim_{t \rightarrow 0} \frac{f(x + tu) - f(x)}{t} \right)$$

for $x \in \mathbb{V}$. Fixing a basis $\mathbf{v} = \{v_1, v_2, \dots, v_d\}$ of \mathbb{V} , we introduce, for each multi-index $\beta \in \mathbb{N}^d$, $D_{\mathbf{v}}^\beta = (D_{v_1})^{\beta_1} (D_{v_2})^{\beta_2} \dots (D_{v_d})^{\beta_d}$.

Proposition 2.5. *Let Λ be a nondegenerate-homogeneous operator on \mathbb{V} . Then there exist a basis $\mathbf{v} = \{v_1, v_2, \dots, v_d\}$ of \mathbb{V} and $\mathbf{n} = (n_1, n_2, \dots, n_d) \in \mathbb{N}_+^d$ for which*

$$\Lambda = \sum_{|\beta:\mathbf{n}|=1} a_\beta D_{\mathbf{v}}^\beta. \quad (5)$$

where $\{a_\beta\} \subseteq \mathbb{C}$. The isomorphism $E_{\mathbf{v}}^{\mathbf{n}} \in Gl(\mathbb{V})$, defined by $E_{\mathbf{v}}^{\mathbf{n}} v_k = (1/n_k) v_k$ for $k = 1, 2, \dots, d$, is a member of $\text{Exp}(\Lambda)$. Further, if Λ is positive-homogeneous, then $\mathbf{n} = 2\mathbf{m}$ for $\mathbf{m} = (m_1, m_2, \dots, m_d) \in \mathbb{N}_+^d$ and hence

$$\Lambda = \sum_{|\beta:\mathbf{m}|=2} a_\beta D_{\mathbf{v}}^\beta.$$

We will sometimes refer to the \mathbf{n} and \mathbf{m} of the proposition as *weights*. Before addressing the proposition, we first prove the following mirrored result for symbols.

Lemma 2.6. *Let P be a nondegenerate-homogeneous polynomial on a d -dimensional real vector space W . Then there exists a basis $\mathbf{w} = \{w_1, w_2, \dots, w_d\}$ of W and $\mathbf{n} = (n_1, n_2, \dots, n_d) \in \mathbb{N}_+^d$ for which*

$$P(\xi) = \sum_{|\beta:\mathbf{n}|=1} a_\beta \xi^\beta$$

for all $\xi = \xi_1 w_1 + \xi_2 w_2 + \dots + \xi_d w_d \in W$ where $\xi^\beta := (\xi_1)^{\beta_1} (\xi_2)^{\beta_2} \dots (\xi_d)^{\beta_d}$ and $\{a_\beta\} \subseteq \mathbb{C}$. The isomorphism $E_{\mathbf{w}}^{\mathbf{n}} \in Gl(\mathbb{V})$, defined by $E_{\mathbf{w}}^{\mathbf{n}} w_k = (1/n_k) w_k$ for $k = 1, 2, \dots, d$, is a member of $\text{Exp}(P)$. Further, if P is a positive-definite polynomial, i.e., it is positive-homogeneous, then $\mathbf{n} = 2\mathbf{m}$ for $\mathbf{m} = (m_1, m_2, \dots, m_d) \in \mathbb{N}_+^d$ and hence

$$P(\xi) = \sum_{|\beta:\mathbf{m}|=2} a_\beta \xi^\beta$$

for $\xi \in W$.

Proof. Let $E \in \text{Exp}(P)$ be diagonalizable and select a basis $\mathbf{w} = \{w_1, w_2, \dots, w_d\}$ which diagonalizes E , i.e., $E w_k = \delta_k w_k$ where $\delta_k \in \mathbb{R}$ for $k = 1, 2, \dots, d$. Because P is a polynomial, there exists a finite collection $\{a_\beta\} \subseteq \mathbb{C}$ for which

$$P(\xi) = \sum_{\beta} a_\beta \xi^\beta$$

for $\xi \in W$. By invoking the homogeneity of P with respect to E and using the fact that $t^E w_k = t^{\delta_k} w_k$ for $k = 1, 2, \dots, d$, we have

$$t \sum_{\beta} a_{\beta} \xi^{\beta} = \sum_{\beta} a_{\beta} (t^E \xi)^{\beta} = \sum_{\beta} a_{\beta} t^{\delta \cdot \beta} \xi^{\beta}$$

for all $\xi \in W$ and $t > 0$ where $\delta \cdot \beta = \delta_1 \beta_1 + \delta_2 \beta_2 + \dots + \delta_d \beta_d$. In view of the nondegenerateness of P , the linear independence of distinct powers of t and the polynomial functions $\xi \mapsto \xi^{\beta}$, for distinct multi-indices β , as C^{∞} functions ensures that $a_{\beta} = 0$ unless $\beta \cdot \delta = 1$. We can therefore write

$$P(\xi) = \sum_{\beta \cdot \delta = 1} a_{\beta} \xi^{\beta} \quad (6)$$

for $\xi \in W$. We now determine $\delta = (\delta_1, \delta_2, \dots, \delta_d)$ by evaluating this polynomial along the coordinate axes. To this end, by fixing $k = 1, 2, \dots, d$ and setting $\xi = x w_k$ for $x \in \mathbb{R}$, it is easy to see that the summation above collapses into a single term $a_{\beta} x^{|\beta|}$ where $\beta = |\beta| e_k = (1/\delta_k) e_k$ (here e_k denotes the usual k th-Euclidean basis vector in \mathbb{R}^d). Consequently, $n_k := 1/\delta_k \in \mathbb{N}_+$ for $k = 1, 2, \dots, d$ and thus, upon setting $\mathbf{n} = (n_1, n_2, \dots, n_d)$, (6) yields

$$P(\xi) = \sum_{|\beta \cdot \mathbf{n}| = 1} a_{\beta} \xi^{\beta}$$

for all $\xi \in W$ as was asserted. In this notation, it is also evident that $E_{\mathbf{w}}^{\mathbf{n}} = E \in \text{Exp}(P)$. Under the additional assumption that P is positive-definite, we again evaluate P at the coordinate axes to see that $\text{Re} P(x w_k) = \text{Re}(a_{n_k e_k}) x^{n_k}$ for $x \in \mathbb{R}$. In this case, the positive-definiteness of P requires $\text{Re}(a_{n_k e_k}) > 0$ and $n_k \in 2\mathbb{N}_+$ for each $k = 1, 2, \dots, d$. Consequently, $\mathbf{n} = 2\mathbf{m}$ for $\mathbf{m} = (m_1, m_2, \dots, m_d) \in \mathbb{N}_+^d$ as desired. \square

Proof of Proposition 2.5. Given a nondegenerate-homogeneous Λ on \mathbb{V} with symbol P , P is necessarily a nondegenerate-homogeneous polynomial on \mathbb{V}^* in view of Proposition 2.3. We can therefore apply Lemma 2.6 to select a basis $\mathbf{v}^* = \{v_1^*, v_2^*, \dots, v_d^*\}$ of \mathbb{V}^* and $\mathbf{n} = (n_1, n_2, \dots, n_d) \in \mathbb{N}_+^d$ for which

$$P(\xi) = \sum_{|\beta \cdot \mathbf{n}| = 1} a_{\beta} \xi^{\beta} \quad (7)$$

for all $\xi = \xi_1 v_1^* + \xi_2 v_2^* + \dots + \xi_d v_d^*$ where $\{a_{\beta}\} \subseteq \mathbb{C}$. We will denote by \mathbf{v} , the dual basis to \mathbf{v}^* , i.e., $\mathbf{v} = \{v_1, v_2, \dots, v_d\}$ is the unique basis of \mathbb{V} for which $v_k^*(v_l) = 1$ when $k = l$ and 0 otherwise. In view of the duality of the bases \mathbf{v} and \mathbf{v}^* , it is straightforward to verify that, for each multi-index β , the symbol of $D_{\mathbb{V}}^{\beta}$ is ξ^{β} in the notation of Lemma 2.6. Consequently, the constant-coefficient partial differential operator defined by the right hand side of (5) also has symbol P and so it must be equal to Λ because operators and symbols are in one-to-one correspondence. Using (5), it is now straightforward to verify that $E_{\mathbb{V}}^{\mathbf{n}} \in \text{Exp}(\Lambda)$. The assertion that $\mathbf{n} = 2\mathbf{m}$ when Λ is positive-homogeneous follows from the analogous conclusion of Lemma 2.6 by the same line of reasoning. \square

In view of Proposition 2.5, we see that all nondegenerate-homogeneous operators are semi-elliptic in some linear coordinate system (that which is defined by \mathbf{v}). An appeal to Theorem 11.1.11 of [46] immediately yields the following corollary.

Corollary 2.7. *Every nondegenerate-homogeneous operator Λ on \mathbb{V} is hypoelliptic.*

Our next goal is to associate an “order” to each nondegenerate-homogeneous operator. For a positive-homogeneous operator Λ , this order will be seen to govern the on-diagonal decay of its heat kernel K_Λ and so, equivalently, the ultracontractivity of the semigroup $e^{-t\Lambda}$. With the help of Lemma 2.6, the few lemmas in this direction come easily.

Lemma 2.8. *Let P be a nondegenerate-homogeneous polynomial on a d -dimensional real vector space W . Then $\lim_{\xi \rightarrow \infty} |P(\xi)| = \infty$; here $\xi \rightarrow \infty$ means that $|\xi| \rightarrow \infty$ in any (and hence every) norm on W .*

Proof. The idea of the proof is to construct a function which bounds $|P|$ from below and obviously blows up at infinity. To this end, let \mathbf{w} be a basis for W and take $\mathbf{n} \in \mathbb{N}_+^d$ as guaranteed by Lemma 2.6; we have $E_{\mathbf{w}}^{\mathbf{n}} \in \text{Exp}(P)$ where $E_{\mathbf{w}}^{\mathbf{n}} w_k = (1/n_k)w_k$ for $k = 1, 2, \dots, d$. Define $|\cdot|_{\mathbf{w}}^{\mathbf{n}} : W \rightarrow [0, \infty)$ by

$$|\xi|_{\mathbf{w}}^{\mathbf{n}} = \sum_{k=1}^d |\xi_k|^n$$

where $\xi = \xi_1 w_1 + \xi_2 w_2 + \dots + \xi_d w_d \in W$. We observe immediately $E_{\mathbf{w}}^{\mathbf{n}} \in \text{Exp}(|\cdot|_{\mathbf{w}}^{\mathbf{n}})$ because $t E_{\mathbf{w}}^{\mathbf{n}} w_k = t^{1/n_k} w_k$ for $k = 1, 2, \dots, d$. An application of Proposition 3.2 (a basic result appearing in our background section, Section 3), which uses the nondegenerateness of P , gives a positive constant C for which $|\xi|_{\mathbf{w}}^{\mathbf{n}} \leq C|P(\xi)|$ for all $\xi \in W$. The lemma now follows by simply noting that $|\xi|_{\mathbf{w}}^{\mathbf{n}} \rightarrow \infty$ as $\xi \rightarrow \infty$. \square

Lemma 2.9. *Let P be a polynomial on W and denote by $\text{Sym}(P)$ the set of $O \in \text{End}(W)$ for which $P(O\xi) = P(\xi)$ for all $\xi \in W$. If P is a nondegenerate-homogeneous polynomial, then $\text{Sym}(P)$, called the symmetry group of P , is a compact subgroup of $\text{Gl}(W)$.*

Proof. Our supposition that P is a nondegenerate polynomial ensures that, for each $O \in \text{Sym}(P)$, $\text{Ker}(O)$ is empty and hence $O \in \text{Gl}(W)$. Consequently, given O_1 and $O_2 \in \text{Sym}(P)$, we observe that $P(O_1^{-1}\xi) = P(O_1 O_1^{-1}\xi) = P(\xi)$ and $P(O_1 O_2 \xi) = P(O_2 \xi) = P(\xi)$ for all $\xi \in W$; therefore $\text{Sym}(P)$ is a subgroup of $\text{Gl}(W)$.

To see that $\text{Sym}(P)$ is compact, in view of the finite-dimensionality of $\text{Gl}(W)$ and the Heine-Borel theorem, it suffices to show that $\text{Sym}(P)$ is closed and bounded. First, for any sequence $\{O_n\} \subseteq \text{Sym}(P)$ for which $O_n \rightarrow O$ as $n \rightarrow \infty$, the continuity of P ensures that $P(O\xi) = \lim_{n \rightarrow \infty} P(O_n \xi) = \lim_{n \rightarrow \infty} P(\xi) = P(\xi)$ for each $\xi \in W$ and therefore $\text{Sym}(P)$ is closed. It remains to show that $\text{Sym}(P)$ is bounded; this is the only piece of the proof that makes use of the fact that P is nondegenerate-homogeneous and not simply homogeneous. Assume that, to reach a contradiction, that there exists an unbounded sequence $\{O_n\} \subseteq \text{Sym}(P)$. Choosing a norm $|\cdot|$ on W , let S be the corresponding unit sphere in W . Then there exists a sequence $\{\xi_n\} \subseteq W$ for which $|\xi_n| = 1$ for all $n \in \mathbb{N}_+$ but $\lim_{n \rightarrow \infty} |O_n \xi_n| = \infty$. In view of Lemma 2.8,

$$\infty = \lim_{n \rightarrow \infty} |P(O_n \xi_n)| = \lim_{n \rightarrow \infty} |P(\xi_n)| \leq \sup_{\xi \in S} |P(\xi)|,$$

which cannot be true for P is necessarily bounded on S because it is continuous. \square

Lemma 2.10. *Let Λ be a nondegenerate-homogeneous operator. For any $E_1, E_2 \in \text{Exp}(\Lambda)$,*

$$\text{tr } E_1 = \text{tr } E_2.$$

Proof. Let P be the symbol of Λ and take $E_1, E_2 \in \text{Exp}(\Lambda)$. Since $E_1^*, E_2^* \in \text{Exp}(P)$, $t^{E_1^*} t^{-E_2^*} \in \text{Sym}(P)$ for all $t > 0$. As $\text{Sym}(P)$ is a compact group in view of the previous lemma, the determinant map $\det : \text{Gl}(\mathbb{V}^*) \rightarrow \mathbb{C}^*$, a Lie group homomorphism, necessarily maps $\text{Sym}(P)$ into the unit circle. Consequently,

$$1 = |\det(t^{E_1^*} t^{-E_2^*})| = |\det(t^{E_1^*}) \det(t^{-E_2^*})| = |t^{\text{tr } E_1^*} t^{-\text{tr } E_2^*}| = t^{\text{tr } E_1^* - \text{tr } E_2^*}$$

for all $t > 0$. Therefore, $\text{tr } E_1 = \text{tr } E_1^* = \text{tr } E_2^* = \text{tr } E_2$ as desired. \square

By the above lemma, to each nondegenerate-homogeneous operator Λ , we define the *homogeneous order* of Λ to be the number

$$\mu_\Lambda = \text{tr } E$$

for any $E \in \text{Exp}(\Lambda)$. By an appeal to Proposition 2.5, $E_\mathbb{V}^{\mathbf{n}} \in \text{Exp}(\Lambda)$ for some $\mathbf{n} \in \mathbb{N}_+$ and so we observe that

$$\mu_\Lambda = \frac{1}{n_1} + \frac{1}{n_2} + \cdots + \frac{1}{n_d}. \quad (8)$$

In particular, μ_Λ is a positive rational number.

2.1 Positive-homogeneous operators and their heat kernels

We now restrict our attention to the study of positive-homogeneous operators and their associated heat kernels. To this end, let Λ be a positive-homogeneous operator on \mathbb{V} with symbol P and homogeneous order μ_Λ . The heat kernel for Λ arises naturally from the study of the following Cauchy problem for the corresponding heat equation $\partial_t + \Lambda = 0$: Given initial data $f : \mathbb{V} \rightarrow \mathbb{C}$ which is, say, bounded and continuous, find $u(t, x)$ satisfying

$$\begin{cases} (\partial_t + \Lambda) u = 0 & \text{in } (0, \infty) \times \mathbb{V} \\ u(0, x) = f(x) & \text{for } x \in \mathbb{V}. \end{cases} \quad (9)$$

The initial value problem (9) is solved by putting

$$u(t, x) = \int_{\mathbb{V}} K_\Lambda^t(x - y) f(y) dy$$

where $K_\Lambda^{(\cdot)}(\cdot) : (0, \infty) \times \mathbb{V} \rightarrow \mathbb{C}$ is defined by

$$K_\Lambda^t(x) = \mathcal{F}^{-1} (e^{-tP}) (x) = \int_{\mathbb{V}^*} e^{-i\xi(x)} e^{-tP(\xi)} d\xi$$

for $t > 0$ and $x \in \mathbb{V}$; we call K_Λ the *heat kernel* associated to Λ . Equivalently, K_Λ is the integral (convolution) kernel of the continuous semigroup $\{e^{-t\Lambda}\}_{t>0}$ of bounded operators on $L^2(\mathbb{V})$ with infinitesimal generator $-\Lambda$. That is, for each $f \in L^2(\mathbb{V})$,

$$(e^{-t\Lambda} f) (x) = \int_{\mathbb{V}} K_\Lambda^t(x - y) f(y) dy \quad (10)$$

for $t > 0$ and $x \in \mathbb{V}$. Let us make some simple observations about K_Λ . First, by virtue of Lemma 2.8, it follows that $K_\Lambda^t \in \mathcal{S}(\mathbb{V})$ for each $t > 0$. Further, for any $E \in \text{Exp}(\Lambda)$,

$$\begin{aligned} K_\Lambda^t(x) &= \int_{\mathbb{V}^*} e^{-i\xi(x)} e^{-P(tE^* \xi)} d\xi \\ &= \int_{\mathbb{V}^*} e^{-i(t^{-E^*})\xi(x)} e^{-P(\xi)} \det(t^{-E^*}) d\xi = \frac{1}{t^{\text{tr } E}} \int_{\mathbb{V}^*} e^{-i\xi(t^{-E}x)} e^{-P(\xi)} d\xi = \frac{1}{t^{\mu_\Lambda}} K_\Lambda^1(t^{-E}x) \end{aligned}$$

for $t > 0$ and $x \in \mathbb{V}$. This computation immediately yields the so-called on-diagonal estimate for K_Λ ,

$$\|e^{-t\Lambda}\|_{1 \rightarrow \infty} = \|K_\Lambda^t\|_\infty = \frac{1}{t^{\mu_\Lambda}} \|K_\Lambda^1\|_\infty \leq \frac{C}{t^{\mu_\Lambda}}$$

for $t > 0$; this is equivalently a statement of ultracontractivity for the semigroup $e^{-t\Lambda}$. As it turns out, we can say something much stronger.

Proposition 2.11. *Let Λ be a positive-homogeneous operator with symbol P and homogeneous order μ_Λ . Let $R^\# : \mathbb{V} \rightarrow \mathbb{R}$ be the Legendre-Fenchel transform of $R = \text{Re } P$ defined by*

$$R^\#(x) = \sup_{\xi \in \mathbb{V}^*} \{\xi(x) - R(\xi)\}$$

for $x \in \mathbb{V}$. Also, let \mathbf{v} and $\mathbf{m} \in \mathbb{N}_+^d$ be as guaranteed by Proposition 2.5. Then, there exist positive constants C_0 and M and, for each multi-index β , a positive constant C_β such that, for all $k \in \mathbb{N}$,

$$\left| \partial_t^k D_{\mathbb{V}}^\beta K_\Lambda^t(x-y) \right| \leq \frac{C_\beta C_0^k k!}{t^{\mu_\Lambda + k + |\beta : 2\mathbf{m}|}} \exp\left(-tMR^\#\left(\frac{x-y}{t}\right)\right) \quad (11)$$

for all $x, y \in \mathbb{V}$ and $t > 0$. In particular,

$$\left| K_\Lambda^t(x-y) \right| \leq \frac{C}{t^{\mu_\Lambda}} \exp\left(-tMR^\#\left(\frac{x-y}{t}\right)\right) \quad (12)$$

for all $x, y \in \mathbb{V}$ and $t > 0$.

Remark 2. In view of (8), the exponent on the prefactor in (11) can be equivalently written, for any multi-index β and $k \in \mathbb{N}$, as $\mu_\Lambda + k + |\beta : 2\mathbf{m}| = k + |\mathbf{1} + \beta : 2\mathbf{m}| = |\mathbf{1} + 2k\mathbf{m} + \beta : 2\mathbf{m}|$ where $\mathbf{1} = (1, 1, \dots, 1)$.

We prove the proposition above in the Section 5; the remainder of this section is dedicated to discussing the result and connecting it to the existing theory. Let us first note that the estimate (11) is mirrored by an analogous space-time estimate, Theorem 5.3 of [53], for the convolution powers of complex-valued functions on \mathbb{Z}^d satisfying certain conditions (see Section 5 of [53]). The relationship between these two results, Theorem 5.3 of [53] and Proposition 2.11, parallels the relationship between Gaussian off-diagonal estimates for random walks and the analogous off-diagonal estimates enjoyed by the classical heat kernel [42].

Let us first show that the estimates (11) and (12) recapture the well-known estimates of the theory of parabolic equations and systems in \mathbb{R}^d – a theory in which the Laplacian operator $\Delta = \sum_{l=1}^d \partial_{x_l}^2$ and its integer powers play a central role. To place things into the context of this article, let us observe that, for each positive integer m , the partial differential operator $(-\Delta)^m$ is a positive-homogeneous operator on \mathbb{R}^d with symbol $P(\xi) = |\xi|^{2m}$; here, we identify \mathbb{R}^d as its own dual equipped with the dot product and Euclidean norm $|\cdot|$. Indeed, one easily observes that $P = |\cdot|^{2m}$ is a positive-definite polynomial and $E = (2m)^{-1}I \in \text{Exp}((-\Delta)^m)$ where $I \in \text{Gl}(\mathbb{R}^d)$ is the identity. Consequently, the homogeneous order of $(-\Delta)^m$ is $d/2m = (2m)^{-1} \text{tr}(I)$ and the Legendre-Fenchel transform of $R = \text{Re}P = |\cdot|^{2m}$ is easily computed to be $R^\#(x) = C_m|x|^{2m/(2m-1)}$ where $C_m = (2m)^{1/(2m-1)} - (2m)^{-2m/(2m-1)} > 0$. Hence, (12) is the well-known estimate

$$\left| K_{(-\Delta)^m}^t(x-y) \right| \leq \frac{C}{t^{d/2m}} \exp\left(-M \frac{|x-y|^{2m/(2m-1)}}{t^{1/(2m-1)}}\right)$$

for $x, y \in \mathbb{R}^d$ and $t > 0$; this so-called off-diagonal estimate is ubiquitous to the theory of “higher-order” elliptic and parabolic equations [16, 35, 39, 54]. To write the derivative estimate (11) in this context, we first observe that the basis given by Proposition 2.5 can be taken to be the standard Euclidean basis, $\mathbf{e} = \{e_1, e_2, \dots, e_d\}$ and further, $\mathbf{m} = (m, m, \dots, m)$ is the (isotropic) weight given by the proposition. Writing $D^\beta = D_{\mathbf{e}}^\beta = (i\partial_{x_1})^{\beta_1} (i\partial_{x_2})^{\beta_2} \dots (i\partial_{x_d})^{\beta_d}$ and $|\beta| = \beta_1 + \beta_2 + \dots + \beta_d$ for each multi-index β , (11) takes the form

$$\left| \partial_t^k D^\beta K_{(-\Delta)^m}^t(x-y) \right| \leq \frac{C}{t^{(d+|\beta|)/2m+k}} \exp\left(-M \frac{|x-y|^{2m/(2m-1)}}{t^{1/(2m-1)}}\right)$$

for $x, y \in \mathbb{R}^d$ and $t > 0$, c.f., [35, Property 4, p. 93].

The appearance of the 1-dimensional Legendre-Fenchel transform in heat kernel estimates was previously recognized and exploited in [8] and [9] in the context of elliptic operators. Due to the isotropic nature of elliptic operators, the 1-dimensional transform is sufficient to capture the inherent isotropic decay of corresponding heat kernels. Beyond the elliptic theory, the appearance of the full d -dimensional Legendre-Fenchel transform is remarkable because it sharply captures the general anisotropic decay of K_Λ . Consider, for instance, the particularly simple positive-homogeneous operator $\Lambda = -\partial_{x_1}^6 + \partial_{x_2}^8$ on \mathbb{R}^2 with symbol $P(\xi_1, \xi_2) = \xi_1^6 + \xi_2^8$. It is easily checked that the operator E with matrix representation $\text{diag}(1/6, 1/8)$, in the standard Euclidean basis, is a member of the $\text{Exp}(\Lambda)$ and so the homogeneous order of Λ is $\mu_\Lambda = \text{tr}(\text{diag}(1/6, 1/8)) = 7/24$. Here we can compute the Legendre-Fenchel transform of $R = \text{Re}P = P$ directly to obtain $R^\#(x_1, x_2) = c_1|x_1|^{6/5} + c_2|x_2|^{8/7}$ for $(x_1, x_2) \in \mathbb{R}^2$ where c_1 and c_2 are positive constants. In this case, Proposition 2.11 gives positive constants M_1, M_2 and C for which

$$\left| K_\Lambda^t(x_1 - y_1, x_2 - y_2) \right| \leq \frac{C}{t^{7/24}} \exp\left(-\left(M_1 \frac{|x_1 - y_1|^{6/5}}{t^{1/5}} + M_2 \frac{|x_2 - y_2|^{8/7}}{t^{1/7}}\right)\right) \quad (13)$$

for $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$ and $t > 0$. We note however that Λ is “separable” and so we can write $K_\Lambda^t(x_1, x_2) = K_{(-\Delta)^3}^t(x_1)K_{(-\Delta)^4}^t(x_2)$ where Δ is the 1-dimensional Laplacian operator. In view of Theorem 8 of [8] and its subsequent remark, the estimate (13) is seen to be sharp (modulo the values of M_1, M_2 and

C). To further illustrate the proposition for a less simple positive-homogeneous operator, we consider the operator Λ appearing in Example 3. In this case,

$$R(\xi_1, \xi_2) = P(\xi_1, \xi_2) = \frac{1}{8}(\xi_1 + \xi_2)^2 + \frac{23}{384}(\xi_1 - \xi_2)^4$$

and one can verify directly that the $E \in \text{End}(\mathbb{R}^2)$, with matrix representation

$$E_{\mathbf{e}} = \begin{pmatrix} 3/8 & 1/8 \\ 1/8 & 3/8 \end{pmatrix}$$

in the standard Euclidean basis, is a member of $\text{Exp}(\Lambda)$. From this, we immediately obtain $\mu_\Lambda = \text{tr}(E) = 3/4$ and one can directly compute

$$R^\#(x_1, x_2) = M_1|x_1 + x_2|^2 + M_2|x_1 - x_2|^{4/3}$$

for $(x_1, x_2) \in \mathbb{R}^2$ where M_1 and M_2 are positive constants. Consequently,

$$|K_\Lambda^t(x_1 - y_1, x_2 - y_2)| \leq \frac{C}{t^{3/4}} \exp \left(- \left(M_1 \frac{|(x_1 - y_1) + (x_2 - y_2)|^2}{t} + M_2 \frac{|(x_1 - y_1) - (x_2 - y_2)|^{4/3}}{t^{1/3}} \right) \right)$$

for $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$ and $t > 0$. Furthermore, $\mathbf{m} = (1, 2) \in \mathbb{N}_+^2$ and the basis $\mathbf{v} = \{v_1, v_2\}$ of \mathbb{R}^2 given in discussion surrounding (3) are precisely those guaranteed by Proposition 2.5. Appealing to the full strength of Proposition 2.11, we obtain positive constants C, M_1 and M_2 and, for each multi-index β , a positive constant C_β such that, for each $k \in \mathbb{N}$,

$$\begin{aligned} & \left| \partial_t^k D_{\mathbf{v}}^\beta K_\Lambda(x_1 - y_1, x_2 - y_2) \right| \\ & \leq \frac{C_\beta C_0^k k!}{t^{3/4+k+|\beta:2\mathbf{m}|}} \exp \left(- \left(M_1 \frac{|(x_1 - y_1) + (x_2 - y_2)|^2}{t} + M_2 \frac{|(x_1 - y_1) - (x_2 - y_2)|^{4/3}}{t^{1/3}} \right) \right) \end{aligned}$$

for $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$ and $t > 0$.

In the context of homogeneous groups, the off-diagonal behavior for the heat kernel of a positive Rockland operator (a positive self-adjoint operator which is homogeneous with respect to the fixed dilation structure) has been studied in [7, 34, 41] (see also [5]). Given a positive Rockland operator Λ on homogeneous group G , the best known estimate for the heat kernel K_Λ , due to Auscher, ter Elst and Robinson, is of the form

$$|K_\Lambda^t(h^{-1}g)| \leq \frac{C}{t^{\mu_\Lambda}} \exp \left(-M \left(\frac{\|h^{-1}g\|^{2m}}{t} \right)^{1/(2m-1)} \right) \quad (14)$$

where $\|\cdot\|$ is a homogeneous norm on G (consistent with Λ) and $2m$ is the highest order derivative appearing in Λ . In the context of \mathbb{R}^d , given a symmetric and positive-homogeneous operator Λ with symbol P , the structure $G_D = (\mathbb{R}^d, \{\delta_t^D\})$ for $D = 2mE$ where $E \in \text{Exp}(\Lambda)$ is a homogeneous group on which Λ becomes a positive Rockland operator. On G_D , it is quickly verified that $\|\cdot\| = R(\cdot)^{1/2m}$ is a homogeneous

norm (consistent with Λ) and so the above estimate is given in terms of $R(\cdot)^{1/(2m-1)}$ which is, in general, dominated by the Legendre-Fenchel transform of R . To see this, we need not look further than our previous and simple example in which $\Lambda = -\partial_{x_1}^6 + \partial_{x_2}^8$. Here $2m = 8$ and so $R(x_1, x_2)^{1/(2m-1)} = (|x_1|^6 + |x_2|^8)^{1/7}$. In view of (13), the estimate (14) gives the correct decay along the x_2 -coordinate axis; however, the bounds decay at markedly different rates along the x_1 -coordinate axis. This illustrates that the estimate (14) is suboptimal, at least in the context of \mathbb{R}^d , and thus leads to the natural question: For positive-homogeneous operators on a general homogeneous group G , what is to replace the Legendre-Fenchel transform in heat kernel estimates?

Returning to the general picture, let Λ be a positive-homogeneous operator on \mathbb{V} with symbol P and homogeneous order μ_Λ . To highlight some remarkable properties about the estimates (11) and (12) in this general setting, the following proposition concerning $R^\#$ is useful; for a proof, see Section 8.3 of [53].

Proposition 2.12. *Let Λ be a positive-homogeneous operator with symbol P and let $R^\#$ be the Legendre-Fenchel transform of $R = \text{Re}P$. Then, for any $E \in \text{Exp}(\Lambda)$, $I - E \in \text{Exp}(R^\#)$. Moreover $R^\#$ is continuous, positive-definite in the sense that $R^\#(x) \geq 0$ and $R^\#(x) = 0$ only when $x = 0$. Further, $R^\#$ grows superlinearly in the sense that, for any norm $|\cdot|$ on \mathbb{V} ,*

$$\lim_{x \rightarrow \infty} \frac{|x|}{R^\#(x)} = 0;$$

in particular, $R^\#(x) \rightarrow \infty$ as $x \rightarrow \infty$.

Let us first note that, in view of the proposition, we can easily rewrite (12), for any $E \in \text{Exp}(\Lambda)$, as

$$|K_\Lambda^t(x - y)| \leq \frac{C}{t^{\mu_\Lambda}} \exp\left(-MR^\#(t^{-E}(x - y))\right)$$

for $x, y \in \mathbb{V}$ and $t > 0$; the analogous rewriting is true for (11). The fact that $R^\#$ is positive-definite and grows superlinearly ensures that the convolution operator $e^{-t\Lambda}$ defined by (10) for $t > 0$ is a bounded operator from L^p to L^q for any $1 \leq p, q \leq \infty$. Of course, we already knew this because K_Λ^t is a Schwartz function; however, when replacing Λ with a variable-coefficient operator H , as we will do in the sections to follow, the validity of the estimate (12) for the kernel of the semigroup $\{e^{-tH}\}$ initially defined on L^2 , guarantees that the semigroup extends to a strongly continuous semigroup $\{e^{-tH_p}\}$ on $L^p(\mathbb{R}^d)$ for all $1 \leq p \leq \infty$ and, what's more, the respective infinitesimal generators $-H_p$ have spectra independent of p [17]. Further, the estimate (12) is key to establishing the boundedness of the Riesz transform, it is connected to the resolution of Kato's square root problem and it provides the appropriate starting point for uniqueness classes of solutions to $\partial_t + H = 0$ [6, 51]. With this motivation in mind, following some background in Section 3, we introduce a class of variable-coefficient operators in Section 4 called $(2\mathbf{m}, \mathbf{v})$ -positive-semi-elliptic operators, each such operator H comparable to a fixed positive-homogeneous operator. In Section 5, under the assumption that H has Hölder continuous coefficients and this notion of comparability is uniform, we construct a fundamental solution to the heat equation $\partial_t + H = 0$ and show the essential role played by the Legendre-Fenchel transform in this construction. As mentioned previously, in a forthcoming work we will study the semigroup $\{e^{-tH}\}$ where H is a divergence-form operator, which is comparable to a fixed positive-homogeneous operator, whose coefficients are at worst measurable. As

the Legendre-Fenchel transform appears here by a complex change of variables followed by a minimization argument, in the measurable coefficient setting it appears quite naturally by an application of the so-called Davies' method, suitably adapted to the positive-homogeneous setting.

3 Contracting groups, Hölder continuity and the Legendre-Fenchel transform

In this section, we provide the necessary background on one-parameter contracting groups, anisotropic Hölder continuity, and the Legendre-Fenchel transform and its interplay with the two previous notions.

3.1 One-parameter contracting groups

In what follows, W is a d -dimensional real vector space with a norm $|\cdot|$; the corresponding operator norm on $\text{Gl}(W)$ is denoted by $\|\cdot\|$. Of course, since everything is finite-dimensional, the usual topologies on W and $\text{Gl}(W)$ are insensitive to the specific choice of norms.

Definition 3.1. Let $\{T_t\}_{t>0} \subseteq \text{Gl}(W)$ be a continuous one-parameter group. $\{T_t\}$ is said to be contracting if

$$\lim_{t \rightarrow 0} \|T_t\| = 0.$$

We easily observe that, for any diagonalizable $E \in \text{End}(W)$ with strictly positive spectrum, the corresponding one-parameter group $\{t^E\}_{t>0}$ is contracting. Indeed, if there exists a basis $\mathbf{w} = \{w_1, w_2, \dots, w_d\}$ of W and a collection of positive numbers $\lambda_1, \lambda_2, \dots, \lambda_d$ for which $Ew_k = \lambda_k w_k$ for $k = 1, 2, \dots, d$, then the one parameter group $\{t^E\}_{t>0}$ has $t^E w_k = t^{\lambda_k} w_k$ for $k = 1, 2, \dots, d$ and $t > 0$. It then follows immediately that $\{t^E\}$ is contracting.

Proposition 3.2. Let Q and R be continuous real-valued functions on W . If $R(w) > 0$ for all $w \neq 0$ and there exists $E \in \text{Exp}(Q) \cap \text{Exp}(R)$ for which $\{t^E\}$ is contracting, then, for some positive constant C , $Q(w) \leq CR(w)$ for all $w \in W$. If additionally $Q(w) > 0$ for all $w \neq 0$, then $Q \asymp R$.

Proof. Let S denote the unit sphere in W and observe that

$$\sup_{w \in S} \frac{Q(w)}{R(w)} =: C < \infty$$

because Q and R are continuous and R is non-zero on S . Now, for any non-zero $w \in W$, the fact that t^E is contracting implies that $t^E w \in S$ for some $t > 0$ by virtue of the intermediate value theorem. Therefore, $Q(w) = Q(t^E w)/t \leq CR(t^E w)/t = CR(w)$. In view of the continuity of Q and R , this inequality must hold for all $w \in W$. When additionally $Q(w) > 0$ for all non-zero w , the conclusion that $Q \asymp R$ is obtained by reversing the roles of Q and R in the preceding argument. \square

Corollary 3.3. Let Λ be a positive-homogeneous operator on \mathbb{V} with symbol P and let $R^\#$ be the Legendre-Fenchel transform of $R = \text{Re } P$. Then, for any positive constant M , $R^\# \asymp (MR)^\#$.

Proof. By virtue of Proposition 2.5, let $\mathbf{m} \in \mathbb{N}_+^d$ and \mathbf{v} be a basis for \mathbb{V} and for which $E_{\mathbf{v}}^{2\mathbf{m}} \in \text{Exp}(\Lambda)$. In view of Proposition 2.12, $R^\#$ and $(MR)^\#$ are both continuous, positive-definite and have $F_{\mathbf{v}}^{2\mathbf{m}} := I - E_{\mathbf{v}}^{2\mathbf{m}} \in \text{Exp}(R^\#) \cap \text{Exp}((MR)^\#)$. Upon noting that $F_{\mathbf{v}}^{2\mathbf{m}}v_k = ((2m_k - 1)/2m_k)v_k$ for $k = 1, 2, \dots, d$, we immediately conclude that $\{t^{F_{\mathbf{v}}^{2\mathbf{m}}}\}$ is contracting and so the corollary follows directly from Proposition 3.2. \square

Lemma 3.4. *Let P be a positive-homogeneous polynomial on W and let $\mathbf{n} = 2\mathbf{m} \in \mathbb{N}_+^d$ and \mathbf{w} be a basis for W for which the conclusion of Lemma 2.6 holds. Let $R = \text{Re } P$ and let β and γ be multi-indices such that $\beta \leq \gamma$ (in the standard partial ordering of multi-indices); we shall assume the notation of the lemma.*

1. *For any $n \in \mathbb{N}_+$ such that $|\beta : \mathbf{m}| \leq 2n$, there exist positive constants M and M' for which*

$$|\xi^\gamma \nu^{\beta-\gamma}| \leq M(R(\xi) + R(\nu))^n + M'$$

for all $\xi, \nu \in W$.

2. *If $|\beta : \mathbf{m}| = 2$, there exist positive constants M and M' for which*

$$|\xi^\gamma \nu^{\beta-\gamma}| \leq MR(\xi) + M'R(\nu)$$

for all $\nu, \xi \in W$.

3. *If $|\beta : \mathbf{m}| = 2$ and $\beta > \gamma$, then for every $\epsilon > 0$ there exists a positive constant M for which*

$$|\xi^\gamma \nu^{\beta-\gamma}| \leq \epsilon R(\xi) + MR(\nu)$$

for all $\nu, \xi \in W$.

Proof. Assuming the notation of Lemma 2.6, let $E = E_{\mathbf{w}}^{2\mathbf{m}} \in \text{End}(W)$ and consider the contracting group $\{t^{E \oplus E}\} = \{t^E \oplus t^E\}$ on $W \oplus W$. Because R is a positive-definite polynomial, it immediately follows that $W \oplus W \ni (\xi, \nu) \mapsto R(\xi) + R(\nu)$ is positive-definite. Let $|\cdot|$ be a norm on $W \oplus W$ and respectively denote by B and S the corresponding unit ball and unit sphere in this norm.

To see Item 1, first observe that

$$\sup_{(\xi, \nu) \in S} \frac{|\xi^\gamma \nu^{\beta-\gamma}|}{(R(\xi) + R(\nu))^n} =: M < \infty$$

Now, for any $(\xi, \nu) \in W \oplus W \setminus B$, because $\{t^{E \oplus E}\}$ is contracting, it follows from the intermediate value theorem that, for some $t \geq 1$, $t^{-(E \oplus E)}(\xi, \nu) = (t^{-E}\xi, t^{-E}\nu) \in S$. Correspondingly,

$$\begin{aligned} |\xi^\gamma \nu^{\beta-\gamma}| &= t^{|\beta:2\mathbf{m}|} |(t^{-E}\xi)^\gamma (t^{-E}\nu)^{\beta-\gamma}| \\ &\leq t^{|\beta:2\mathbf{m}|} M (R(t^{-E}\xi) + R(t^{-E}\nu))^n \\ &\leq t^{|\beta:\mathbf{m}|/2-n} M (R(\xi) + R(\nu))^n \\ &\leq M (R(\xi) + R(\nu))^n \end{aligned}$$

because $|\beta : \mathbf{m}|/2 \leq n$. One obtains the constant M' and hence the desired inequality by simply noting that $|\xi^\gamma \nu^{\beta-\gamma}|$ is bounded for all $(\xi, \nu) \in B$.

For Item 2, we use analogous reasoning to obtain a positive constant M for which $|\xi^\gamma \nu^{\beta-\gamma}| \leq M(R(\xi) + R(\nu))$ for all $(\xi, \nu) \in S$. Now, for any non-zero $(\xi, \nu) \in W \oplus W$, the intermediate value theorem gives $t > 0$ for which $t^{E \oplus E}(\xi, \nu) = (t^E \xi, t^E \nu) \in S$ and hence

$$|\xi^\gamma \nu^{\beta-\gamma}| \leq t^{-|\beta:2\mathbf{m}|} M(R(t^E \xi) + R(t^E \nu)) = M(R(\xi) + R(\nu))$$

where we have used the fact that $|\beta:2\mathbf{m}| = |\beta:\mathbf{m}|/2 = 1$ and that $E \in \text{Exp}(R)$. As this inequality must also trivially hold at the origin, we can conclude that it holds for all $\xi, \nu \in W$, as desired.

Finally, we prove Item 3. By virtue of Item 2, for any $\xi, \nu \in W$ and $t > 0$,

$$\begin{aligned} |\xi^\gamma \nu^{\beta-\gamma}| &= |(t^E t^{-E} \xi)^\gamma \nu^{\beta-\gamma}| = t^{|\gamma:2\mathbf{m}|} |(t^{-E} \xi)^\gamma \nu^{\beta-\gamma}| \\ &\leq t^{|\gamma:2\mathbf{m}|} (MR(t^{-E} \xi) + M'R(\nu)) = Mt^{|\gamma:2\mathbf{m}|-1} R(\xi) + M't^{|\gamma:2\mathbf{m}|} R(\nu). \end{aligned}$$

Noting that $|\gamma:2\mathbf{m}| - 1 < 0$ because $\gamma < \beta$, we can make the coefficient of $R(\xi)$ arbitrarily small by choosing t sufficiently large and thereby obtaining the desired result. \square

3.2 Notions of regularity and Hölder continuity

Throughout the remainder of this article, \mathbf{v} will denote a fixed basis for \mathbb{V} and correspondingly we henceforth assume the notational conventions appearing in Proposition 2.5 and $\mathbf{n} = 2\mathbf{m}$ is fixed. For $\alpha \in \mathbb{R}_+^d$, consider the homogeneous norm $|\cdot|_{\mathbf{v}}^\alpha$ defined by

$$|x|_{\mathbf{v}}^\alpha = \sum_{i=1}^d |x_i|^{\alpha_i}$$

for $x \in \mathbb{V}$ where $\phi_{\mathbf{v}}(x) = (x_1, x_2, \dots, x_d)$. As one can easily check,

$$|t^{A_\alpha} x|_{\mathbf{v}}^\alpha = t|x|_{\mathbf{v}}^\alpha$$

for all $t > 0$ and $x \in \mathbb{V}$ where $A_\alpha \in \text{End}(\mathbb{V})$ is represented by the matrix

$$(A_\alpha)_{\mathbf{v}} = \text{diag}(\alpha_1^{-1}, \alpha_2^{-1}, \dots, \alpha_d^{-1})$$

with respect to the basis \mathbf{v} .

Definition 3.5. Let $\mathbf{m} \in \mathbb{N}_+^d$. We say that $\alpha \in \mathbb{R}_+^d$ is consistent with \mathbf{m} if

$$A_\alpha = \omega(I - E) \tag{15}$$

for some $\omega > 0$ where A_α is as above and E is that which appears in Proposition 2.5.

As one can check, α is consistent with \mathbf{m} if and only if $\alpha = a^{-1}\omega$ where

$$\omega = \left(\frac{2m_1}{2m_1 - 1}, \frac{2m_2}{2m_2 - 1}, \dots, \frac{2m_d}{2m_d - 1} \right). \tag{16}$$

Definition 3.6. Let $\Omega \subseteq \Omega' \subseteq \mathbb{V}$ and let $f : \Omega' \rightarrow \mathbb{C}$. We say that f is \mathbf{v} -Hölder continuous on Ω if for some $\alpha \in \mathbb{I}_+^d$ and positive constant M ,

$$|f(x) - f(y)| \leq M|x - y|_{\mathbb{V}}^{\alpha} \quad (17)$$

for all $x, y \in \Omega$. In this case we will say that α is the \mathbf{v} -Hölder exponent of f . If $\Omega = \Omega'$ we will simply say that f is \mathbf{v} -Hölder continuous with exponent α .

The following proposition essentially states that, for bounded functions, Hölder continuity is a local property; its proof is straightforward and is omitted.

Proposition 3.7. Let $\Omega \subseteq \mathbb{V}$ be open and non-empty. If f is bounded and \mathbf{v} -Hölder continuous of order $\alpha \in \mathbb{I}_+^d$, then, for any $\beta < \alpha$, f is also \mathbf{v} -Hölder continuous of order β .

In view of the proposition, we immediately obtain the following corollary.

Corollary 3.8. Let $\Omega \subseteq \mathbb{V}$ be open and non-empty and $\mathbf{m} \in \mathbb{N}_+^d$. If f is bounded and \mathbf{v} -Hölder continuous on Ω of order $\beta \in \mathbb{I}_+^d$, there exists $\alpha \in \mathbb{I}_+^d$ which is consistent with \mathbf{m} for which f is also \mathbf{v} -Hölder continuous of order α .

Proof. The statement follows from the proposition by choosing any α , consistent with \mathbf{m} , such that $\alpha \leq \beta$. \square

The following definition captures the minimal regularity we will require of fundamental solutions to the heat equation.

Definition 3.9. Let $\mathbf{n} \in \mathbb{N}_+^d$, \mathbf{v} be a basis of \mathbb{V} and let \mathcal{O} be a non-empty open subset of $[0, T] \times \mathbb{V}$. A function $u(t, x)$ is said to be (\mathbf{n}, \mathbf{v}) -regular on \mathcal{O} if on \mathcal{O} it is continuously differentiable in t and has continuous (spatial) partial derivatives $D_{\mathbb{V}}^{\beta} u(t, x)$ for all multi-indices β for which $|\beta : \mathbf{n}| \leq 1$.

3.3 The Legendre-Fenchel transform and its interplay with \mathbf{v} -Hölder continuity

Throughout this section, R is the real part of the symbol P of a positive-homogeneous operator Λ on \mathbb{V} . We assume the notation of Proposition 2.12 (and hence Proposition 2.5) and write $E = E_{\mathbb{V}}^{2\mathbf{m}}$. Let us first record two important results which follow essentially from Proposition 2.12.

Corollary 3.10.

$$R^{\#} \asymp |\cdot|_{\mathbb{V}}^{\omega}.$$

where ω was defined in (16).

Proof. In view of Propositions 2.5 and 2.12, $F_{\mathbb{V}}^{2\mathbf{m}} = I - E_{\mathbb{V}}^{2\mathbf{m}} \in \text{Exp}(R^{\#}) \cap \text{Exp}(|\cdot|_{\mathbb{V}}^{\omega})$. After recalling that $\{tF_{\mathbb{V}}^{2\mathbf{m}}\}$ is contracting, Proposition 3.2 yields the desired result immediately. \square

By virtue of Proposition 2.12, standard arguments immediately yield the following corollary.

Corollary 3.11. For any $\epsilon > 0$ and polynomial $Q : \mathbb{V} \rightarrow \mathbb{C}$, i.e., Q is a polynomial in any coordinate system, then

$$Q(\cdot)e^{-\epsilon R^{\#}(\cdot)} \in L^{\infty}(\mathbb{V}) \cap L^1(\mathbb{V}).$$

Lemma 3.12. *Let $\gamma = (2m_{\max} - 1)^{-1}$. Then for any $T > 0$, there exists $M > 0$ such that*

$$R^\#(x) \leq Mt^\gamma R^\#(t^{-E}x)$$

for all $x \in \mathbb{V}$ and $0 < t \leq T$.

Proof. In view of Corollary 3.10, it suffices to prove the statement

$$|t^E x|_{\mathbb{V}}^\omega \leq Mt^\gamma |x|_{\mathbb{V}}^\omega$$

for all $x \in \mathbb{V}$ and $0 < t \leq T$ where $M > 0$ and ω is given by (16). But for any $0 < t \leq T$ and $x \in \mathbb{V}$,

$$|t^E x|_{\mathbb{V}}^\omega = \sum_{j=1}^d t^{1/(2m_j-1)} |x_j|^{\omega_j} \leq t^\gamma \sum_{j=1}^d T^{(1/(2m_j-1)-\gamma)} |x_j|^{\omega_j}$$

from which the result follows. □

Lemma 3.13. *Let $\alpha \in \mathbb{I}_+^d$ be consistent with \mathbf{m} . Then there exists positive constants σ and θ such that $0 < \sigma < 1$ and for any $T > 0$ there exists $M > 0$ such that*

$$|x|_{\mathbb{V}}^\alpha \leq Mt^\sigma (R^\#(t^{-E}x))^\theta$$

for all $x \in \mathbb{V}$ and $0 < t \leq T$.

Proof. By an appeal to Corollary 3.10 and Lemma 3.12,

$$|x|_{\mathbb{V}}^\omega \leq Mt^\gamma R^\#(t^{-E}x)$$

for all $x \in \mathbb{V}$ and $0 < t \leq T$. Since α is consistent with \mathbf{m} , $\alpha = a^{-1}\omega$ where a is that of Definition 3.5, the desired inequality follows by setting $\sigma = \gamma/a$ and $\theta = 1/a$. Because $\alpha \in \mathbb{I}_+^d$, it is necessary that $a \geq 2m_{\min}/(2m_{\min} - 1)$ whence $0 < \sigma \leq (2m_{\min} - 1)/(2m_{\min}(2m_{\max} - 1)) < 1$. □

The following corollary is an immediate application of Lemma 3.13.

Corollary 3.14. *Let $f : \mathbb{V} \rightarrow \mathbb{C}$ be \mathbf{v} -Hölder continuous with exponent $\alpha \in \mathbb{I}_+^d$ and suppose that α is consistent with \mathbf{m} . Then there exist positive constants σ and θ such that $0 < \sigma < 1$ and, for any $T > 0$, there exists $M > 0$ such that*

$$|f(x) - f(y)| \leq Mt^\sigma (R^\#(t^{-E}))^\theta$$

for all $x, y \in \mathbb{V}$ and $0 < t \leq T$. In particular, this estimate holds for the coefficients of H .

4 On $(2\mathbf{m}, \mathbf{v})$ -positive-semi-elliptic operators

In this section, we introduce a class of variable-coefficient operators on \mathbb{V} whose heat equations are studied in the next section. These operators, in view of Proposition 2.5, generalize the class of positive-homogeneous operators. Fix a basis \mathbf{v} of \mathbb{V} , $\mathbf{m} \in \mathbb{N}_+^d$ and, in the notation of the previous section, consider a differential operator H of the form

$$\begin{aligned} H &= \sum_{|\beta:\mathbf{m}| \leq 2} a_\beta(x) D_{\mathbf{v}}^\beta = \sum_{|\beta:\mathbf{m}|=2} a_\beta(x) D_{\mathbf{v}}^\beta + \sum_{|\beta:\mathbf{m}| < 2} a_\beta(x) D_{\mathbf{v}}^\beta \\ &:= H_p + H_l \end{aligned}$$

where the coefficients $a_\beta : \mathbb{V} \rightarrow \mathbb{C}$ are bounded functions. The symbol of H , $P : \mathbb{V} \times \mathbb{V}^* \rightarrow \mathbb{C}$, is defined by

$$\begin{aligned} P(y, \xi) &= \sum_{|\beta:\mathbf{m}| \leq 2} a_\beta(y) \xi^\beta = \sum_{|\beta:\mathbf{m}|=2} a_\beta(y) \xi^\beta + \sum_{|\beta:\mathbf{m}| < 2} a_\beta(y) \xi^\beta \\ &:= P_p(y, \xi) + P_l(y, \xi). \end{aligned}$$

for $y \in \mathbb{V}$ and $\xi \in \mathbb{V}^*$. We shall call H_p the principal part of H and correspondingly, P_p is its principal symbol. Let's also define $R : \mathbb{V}^* \rightarrow \mathbb{R}$ by

$$R(\xi) = \operatorname{Re} P_p(0, \xi) \quad (18)$$

for $\xi \in \mathbb{V}^*$. At times, we will freeze the coefficients of H and H_p at a point $y \in \mathbb{V}$ and consider the constant-coefficient operators they define, namely $H(y)$ and $H_p(y)$ (defined in the obvious way). We note that, for each $y \in \mathbb{V}$, $H_p(y)$ is homogeneous with respect to the one-parameter group $\{\delta_t^E\}_{t>0}$ where $E \in \operatorname{Gl}(\mathbb{V})$ is defined by its matrix representation

$$E_{\mathbf{v}} = \operatorname{diag}\{(2m_1)^{-1}, (2m_2)^{-1}, \dots, (2m_d)^{-1}\}$$

in the basis \mathbf{v} ; i.e., it is homogeneous with respect to the same one-parameter group of dilations at each point in space. This also allows us to uniquely define the *homogeneous order of H* by

$$\mu_H = \operatorname{tr} E = (2m_1)^{-1} + (2m_2)^{-1} + \dots + (2m_d)^{-1}. \quad (19)$$

As in the constant-coefficient setting, $H_p(y)$ is not necessarily homogeneous with respect to a unique group of dilations, i.e., it is possible that $\operatorname{Exp}(H_p(y))$ contains members of $\operatorname{Gl}(\mathbb{V})$ distinct from E . However, we shall henceforth only work with the endomorphism E , defined above, for worrying about this non-uniqueness of dilations does not aid our understanding nor will it sharpen our results. Let us further observe that, for each $y \in \mathbb{V}$, $P_p(y, \cdot)$ and R are homogeneous with respect to $\{t^{E^*}\}_{t>0}$ where $E^* \in \operatorname{Gl}(\mathbb{V}^*)$.

Definition 4.1. *The operator H is called $(2\mathbf{m}, \mathbf{v})$ -positive-semi-elliptic if for all $y \in \mathbb{V}$, $\operatorname{Re} P_p(y, \cdot)$ is a positive-definite polynomial. H is called uniformly $(2\mathbf{m}, \mathbf{v})$ -positive-semi-elliptic if it is $(2\mathbf{m}, \mathbf{v})$ -positive-semi-elliptic and there exists $\delta > 0$ for which*

$$\operatorname{Re} P_p(y, \xi) \geq \delta R(\xi)$$

for all $y \in \mathbb{V}$ and $\xi \in \mathbb{V}^*$. When the context is clear, we will simply say that H is positive-semi-elliptic and uniformly positive-semi-elliptic respectively.

In light of the above definition, a semi-elliptic operator H is one that, at every point $y \in \mathbb{V}$, its frozen-coefficient principal part $H_p(y)$, is a constant-coefficient positive-homogeneous operator which is homogeneous with respect to the same one-parameter group of dilations on \mathbb{V} . A uniformly positive-semi-elliptic operator is one that is semi-elliptic and is uniformly comparable to a constant-coefficient positive-homogeneous operator, namely $H_p(0)$. In this way, positive-homogeneous operators take a central role in this theory.

Remark 3. In view of Proposition 2.5, the definition of R via (18) agrees with that we have given for constant-coefficient positive-homogeneous operators.

Remark 4. For an $(2\mathbf{m}, \mathbf{v})$ -positive-semi-elliptic operator H , uniform semi-ellipticity can be formulated in terms of $\operatorname{Re} P_p(y_0, \cdot)$ for any $y_0 \in \mathbb{V}$; such a notion is equivalent in view of Proposition 3.2.

5 The heat equation

For a uniformly positive-semi-elliptic operator H , we are interested in constructing a fundamental solution to the heat equation,

$$(\partial_t + H)u = 0 \tag{20}$$

on the cylinder $[0, T] \times \mathbb{V}$; here and throughout $T > 0$ is arbitrary but fixed. By definition, a fundamental solution to (20) on $[0, T] \times \mathbb{V}$ is a function $Z : (0, T] \times \mathbb{V} \times \mathbb{V} \rightarrow \mathbb{C}$ satisfying the following two properties:

1. For each $y \in \mathbb{V}$, $Z(\cdot, \cdot, y)$ is $(2\mathbf{m}, \mathbf{v})$ -regular on $(0, T) \times \mathbb{V}$ and satisfies (20).
2. For each $f \in C_b(\mathbb{V})$,

$$\lim_{t \downarrow 0} \int_{\mathbb{V}} Z(t, x, y) f(y) dy = f(x)$$

for all $x \in \mathbb{V}$.

Given a fundamental solution Z to (20), one can easily solve the Cauchy problem: Given $f \in C_b(\mathbb{V})$, find $u(t, x)$ satisfying

$$\begin{cases} (\partial_t + H)u = 0 & \text{on } (0, T) \times \mathbb{V} \\ u(0, x) = f(x) & \text{for } x \in \mathbb{V}. \end{cases}$$

This is, of course, solved by putting

$$u(t, x) = \int_{\mathbb{V}} Z(t, x, y) f(y) dy$$

for $x \in \mathbb{V}$ and $0 < t \leq T$ and interpreting $u(0, x)$ as that defined by the limit of $u(t, x)$ as $t \downarrow 0$. The remainder of this paper is essentially dedicated to establishing the following result:

Theorem 5.1. *Let H be uniformly $(2\mathbf{m}, \mathbf{v})$ -positive-semi-elliptic with bounded \mathbf{v} -Hölder continuous coefficients. Let R and μ_H be defined by (18) and (19) respectively and denote by $R^\#$ the Legendre-Fenchel*

transform of R . Then, for any $T > 0$, there exists a fundamental solution $Z : (0, T] \times \mathbb{V} \times \mathbb{V} \rightarrow \mathbb{C}$ to (20) on $[0, T] \times \mathbb{V}$ such that, for some positive constants C and M ,

$$|Z(t, x, y)| \leq \frac{C}{t^{\mu_H}} \exp\left(-tMR^\# \left(\frac{x-y}{t}\right)\right) \quad (21)$$

for $x, y \in \mathbb{V}$ and $0 < t \leq T$.

We remark that, by definition, the fundamental solution Z given by Theorem 5.1 is $(2\mathbf{m}, \mathbf{v})$ -regular. Thus Z is necessarily continuously differentiable in t and has continuous spatial derivatives of all orders β such that $|\beta : \mathbf{m}| \leq 2$.

As we previously mentioned, the result above is implied by the work of S. D. Eidelman for $2\vec{b}$ -parabolic systems on \mathbb{R}^d (where $\vec{b} = \mathbf{m}$) [36, 37]. Eidelman's systems, of the form (2), are slightly more general than we have considered here, for their coefficients are also allowed to depend on t (but in a uniformly Hölder continuous way). Admitting this t -dependence is a relatively straightforward matter and, for simplicity of presentation, we have not included it (see Remark 5). In this slightly more general situation, stated in \mathbb{R}^d and in which $\mathbf{v} = \mathbf{e}$ is the standard Euclidean basis, Theorem 2.2 (p.79) [37] guarantees the existence of a fundamental solution $Z(t, x, y)$ to (2), which has the same regularity appearing in Theorem 5.1 and satisfies

$$|Z(t, x, y)| \leq \frac{C}{t^{1/(2m_1)+1/(2m_2)+\dots+1/(2m_d)}} \exp\left(-M \sum_{k=1}^d \frac{|x_k - y_k|^{2m_k/(2m_k-1)}}{t^{1/(2m_k-1)}}\right) \quad (22)$$

for $x, y \in \mathbb{R}^d$ and $0 < t \leq T$ where C and M are positive constants. By an appeal to Corollary 3.10, we have $R^\# \asymp |\cdot|_{\mathbb{V}}$ and from this we see that the estimates (21) and (22) are comparable.

In view of Corollary 3.8, the hypothesis of Theorem 5.1 concerning the coefficients of H immediately imply the following a priori stronger condition:

Hypothesis 5.2. *There exists $\alpha \in \mathbb{I}_+^d$ which is consistent with \mathbf{m} and for which the coefficients of H are bounded and \mathbf{v} -Hölder continuous on \mathbb{V} of order α .*

5.1 Levi's Method

In this subsection, we construct a fundamental solution to (20) under only the assumption that H , a uniformly $(2\mathbf{m}, \mathbf{v})$ -positive-semi-elliptic operator, satisfies Hypothesis 5.2. Henceforth, all statements include Hypothesis 5.2 without explicit mention. We follow the famous method of E. E. Levi, c.f., [49] as it was adopted for parabolic systems in [35] and [39]. Although well-known, Levi's method is lengthy and tedious and we will break it into three steps. Let's motivate these steps by first discussing the heuristics of the method.

We start by considering the auxiliary equation

$$\left(\partial_t + \sum_{|\beta:\mathbf{m}|=2} a_\beta(y) D_{\mathbf{v}}^\beta\right) u = (\partial_t + H_p(y)) u = 0 \quad (23)$$

where $y \in \mathbb{V}$ is treated as a parameter. This is the so-called frozen-coefficient heat equation. As one easily checks, for each $y \in \mathbb{V}$,

$$G_p(t, x; y) := \int_{\mathbb{V}^*} e^{-i\xi(x)} e^{-tP_p(y, \xi)} d\xi \quad (x \in \mathbb{V}, t > 0)$$

solves (23). By the uniform semi-ellipticity of H , it is clear that $G_p(t, \cdot; y) \in \mathcal{S}(\mathbb{V})$ for $t > 0$ and $y \in \mathbb{V}$. As we shall see, more is true: G_p is an approximate identity in the sense that

$$\lim_{t \downarrow 0} \int_{\mathbb{V}} G_p(t, x - y; y) f(y) dy = f(x)$$

for all $f \in C_b(\mathbb{V})$. Thus, it is reasonable to seek a fundamental solution to (20) of the form

$$\begin{aligned} Z(t, x, y) &= G_p(t, x - y; y) + \int_0^t \int_{\mathbb{V}} G_p(t - s, x - z; z) \phi(s, z, y) dz ds \\ &= G_p(t, x - y; y) + W(t, x, y) \end{aligned} \quad (24)$$

where ϕ is to be chosen to ensure that the correction term W is $(2\mathbf{m}, \mathbf{v})$ -regular, accounts for the fact that G_p solves (23) but not (20), and is “small enough” as $t \rightarrow 0$ so that the approximate identity aspect of Z is inherited directly from G_p .

Assuming for the moment that W is sufficiently regular, let’s apply the heat operator to (24) with the goal of finding an appropriate ϕ to ensure that Z is a solution to (20). Putting

$$K(t, x, y) = -(\partial_t + H)G_p(t, x - y; y),$$

we have formally,

$$\begin{aligned} (\partial_t + H)Z(t, x, y) &= -K(t, x, y) + (\partial_t + H) \int_0^t \int_{\mathbb{V}} G_p(t - s, x - z; z) \phi(s, z, y) dz ds \\ &= -K(t, x, y) + \lim_{s \uparrow t} \int_{\mathbb{V}} G_p(t - s, x - z; z) \phi(s, z, y) dz \\ &\quad - \int_0^t \int_{\mathbb{V}} -(\partial_t + H)G_p(t - s, x - z; z) \phi(s, z, y) dz ds \\ &= -K(t, x, y) + \phi(t, x, y) - \int_0^t \int_{\mathbb{V}} K(t - s, x, z) \phi(s, z, y) dz ds \end{aligned} \quad (25)$$

where we have made use of Leibniz’ rule and our assertion that G_p is an approximate identity. Thus, for Z to satisfy (20), ϕ must satisfy the integral equation

$$\begin{aligned} K(t, x, y) &= \phi(t, x, y) - \int_0^t \int_{\mathbb{V}} K(t - s, x, z) \phi(s, z, y) dz ds \\ &= \phi(t, x, y) - L(\phi)(t, x, y). \end{aligned} \quad (26)$$

Viewing L as a linear integral operator, (26) is the equation $K = (I - L)\phi$ which has the solution

$$\phi = \sum_{n=0}^{\infty} L^n K \quad (27)$$

provided the series converges in an appropriate sense.

Taking the above as purely formal, our construction will proceed as follows: We first establish estimates for G_p and show that G_p is an approximate identity; this is Step 1. In Step 2, we will define ϕ by (27) and, after deducing some subtle estimates, show that ϕ 's defining series converges whence (26) is satisfied. Finally in Step 3, we will make use of the estimates from Steps 1 and 2 to validate the formal calculation made in (25). Everything will be then pieced together to show that Z , defined by (24), is a fundamental solution to (20). Our entire construction depends on obtaining precise estimates for G_p and for this we will rely heavily on the homogeneity of P_p and the Legendre-Fenchel transform of R .

Remark 5. One can allow the coefficients of H to also depend on t in a uniformly continuous way, and Levi's method pushes through by instead taking G_p as the solution to a frozen-coefficient initial value problem [36,37].

Step 1. Estimates for G_p and its derivatives

The lemma below is a basic building block used in our construction of a fundamental solution to (20) via Levi's method and it makes essential use of the uniform semi-ellipticity of H . We note however that the precise form of the constants obtained, as they depend on k and β , are more detailed than needed for the method to work. Also, the partial differential operators $D_{\mathbf{v}}^{\beta}$ of the lemma are understood to act of the x variable of $G_p(t, x; y)$.

Lemma 5.3. *There exist positive constants M and C_0 and, for each multi-index β , a positive constant C_{β} such that, for any $k \in \mathbb{N}$,*

$$|\partial_t^k D_{\mathbf{v}}^{\beta} G_p(t, x; y)| \leq \frac{C_{\beta} C_0^k k!}{t^{\mu_H + k + |\beta:2\mathbf{m}|}} \exp\left(-tMR^{\#}(x/t)\right) \quad (28)$$

for all $x, y \in \mathbb{V}$ and $t > 0$.

Before proving the lemma, let us note that $tR^{\#}(x/t) = R^{\#}(t^{-E}x)$ for all $t > 0$ and $x \in \mathbb{V}$ in view of Proposition 2.12. Thus the estimate (28) can be written equivalently as

$$|\partial_t^k D_{\mathbf{v}}^{\beta} G_p(t, x; y)| \leq \frac{C_{\beta} C_0^k k!}{t^{\mu_H + k + |\beta:2\mathbf{m}|}} \exp(-MR^{\#}(t^{-E}x)) \quad (29)$$

for $x, y \in \mathbb{V}$ and $t > 0$. We will henceforth use these forms interchangeably and without explicit mention.

Proof. Let us first observe that, for each $x, y \in \mathbb{V}$ and $t > 0$,

$$\begin{aligned} \partial_t^k D_{\mathbb{V}}^\beta G_p(t, x; y) &= \int_{\mathbb{V}^*} (P_p(y, \xi))^k \xi^\beta e^{-i\xi(x)} e^{-tP_p(y, \xi)} d\xi \\ &= \int_{\mathbb{V}^*} (P_p(y, t^{-E^*} \xi))^k (t^{-E^*} \xi)^\beta e^{-i\xi(t^{-E}x)} e^{-P_p(y, \xi)t^{-\text{tr } E}} d\xi \\ &= t^{-\mu_H - k - |\beta:2\mathbf{m}|} \int_{\mathbb{V}^*} (P_p(y, \xi))^k \xi^\beta e^{-i\xi(t^{-E}x)} e^{-P_p(y, \xi)} d\xi \end{aligned}$$

where we have used the homogeneity of P_p with respect to $\{t^{E^*}\}$ and the fact that $\mu_H = \text{tr } E$. Therefore

$$t^{\mu_H + k + |\beta:2\mathbf{m}|} (\partial_t^k D_{\mathbb{V}}^\beta G_p(t, \cdot; y))(t^E x) = \int_{\mathbb{V}^*} (P_p(y, \xi))^k \xi^\beta e^{-i\xi(x)} e^{-P_p(y, \xi)} d\xi \quad (30)$$

for all $x, y \in \mathbb{V}$ and $t > 0$. Thus, to establish (28) (equivalently (29)) it suffices to estimate the right hand side of (30) which is independent of t .

The proof of the desired estimate requires making a complex change of variables and for this reason we will work with the complexification of \mathbb{V}^* , whose members are denoted by $z = \xi - i\nu$ for $\xi, \nu \in \mathbb{V}^*$; this space is isomorphic to \mathbb{C}^d . We claim that there are positive constants C_0, M_1, M_2 and, for each multi-index β , a positive constant C_β such that, for each $k \in \mathbb{N}$,

$$|(P_p(y, \xi - i\nu))^k (\xi - i\nu)^\beta e^{-P_p(y, \xi - i\nu)}| \leq C_\beta C_0^k k! e^{-M_1 R(\xi)} e^{M_2 R(\nu)} \quad (31)$$

for all $\xi, \nu \in \mathbb{V}^*$ and $y \in \mathbb{V}$. Let us first observe that

$$P_p(y, \xi - i\nu) = P_p(y, \xi) + \sum_{|\beta:\mathbf{m}|=2} \sum_{\gamma < \beta} a_{\beta, \gamma} \xi^\gamma (-i\nu)^{\beta - \gamma}$$

for all $z, \nu \in \mathbb{V}^*$ and $y \in \mathbb{V}$, where $a_{\beta, \gamma}$ are bounded functions of y arising from the coefficients of H and the coefficients of the multinomial expansion. By virtue of the uniform semi-ellipticity of H and the boundedness of the coefficients, we have

$$-\text{Re } P_p(y, \xi - i\nu) \leq -\delta R(\xi) + C \sum_{|\beta:\mathbf{m}|=2} \sum_{\gamma < \beta} |\xi^\gamma \nu^{\beta - \gamma}|$$

for all $\xi, \nu \in \mathbb{V}^*$ and $y \in \mathbb{V}$ where C is a positive constant. By applying Lemma 3.4 to each term $|\xi^\gamma \nu^{\beta - \gamma}|$ in the summation, we can find a positive constant M for which the entire summation is bounded above by $\delta/2R(\xi) + MR(\nu)$ for all $\xi, \nu \in \mathbb{V}^*$. By setting $M_1 = \delta/6$, we have

$$-\text{Re } P_p(y, \xi - i\nu) \leq -3M_1 R(\xi) + MR(\nu) \quad (32)$$

for all $\xi, \nu \in \mathbb{V}^*$ and $y \in \mathbb{V}$. By analogous reasoning (making use of item 1 of Lemma 3.4), there exists a positive constant C for which

$$|P_p(y, \xi - i\nu)| \leq C(R(\xi) + R(\nu))$$

for all $\xi, \nu \in \mathbb{V}^*$ and $y \in \mathbb{V}$. Thus, for any $k \in \mathbb{N}$,

$$|P_p(y, \xi - i\nu)|^k \leq \frac{C^k k! (M_1(R(\xi) + R(\nu)))^k}{M_1^k k!} \leq C_0^k k! e^{M_1(R(\xi) + R(\nu))} \quad (33)$$

for all $\xi, \nu \in \mathbb{V}^*$ and $y \in \mathbb{V}$ where $C_0 = C/M_1$. Finally, for each multi-index β , another application of Lemma 3.4 gives $C' > 0$ for which

$$|(\xi - i\nu)^\beta| \leq |\xi^\beta| + |\nu^\beta| + \sum_{0 < \gamma < \beta} c_{\gamma, \beta} |\xi^\gamma \nu^{\beta - \gamma}| \leq C' ((R(\xi) + R(\nu))^n + 1)$$

for all $\xi, \nu \in \mathbb{V}^*$ where $n \in \mathbb{N}$ has been chosen to satisfy $|\beta : 2n\mathbf{m}| < 1$. Consequently, there is a positive constant C_β for which

$$|(\xi - i\nu)^\beta| \leq C_\beta e^{M_1(R(\xi) + R(\nu))} \quad (34)$$

for all $\xi, \nu \in \mathbb{V}^*$. Upon combining (32), (33) and (34), we obtain the inequality

$$\left| P_p(y, \xi - i\nu)^k (\xi - i\nu)^\beta e^{-P_p(y, \xi - i\nu)} \right| \leq C_\beta C_0^k k! e^{-M_1 R(\xi) + (M + 2M_1)R(\nu)}$$

which holds for all $\xi, \nu \in \mathbb{V}^*$ and $y \in \mathbb{V}$. Upon paying careful attention to the way in which our constants were chosen, we observe the claim is established by setting $M_2 = M + 2M_1$.

From the claim above, it follows that, for any $\nu \in \mathbb{V}^*$ and $y \in \mathbb{V}$, the following change of coordinates by means of a \mathbb{C}^d contour integral is justified:

$$\begin{aligned} \int_{\mathbb{V}^*} (P_p(y, \xi))^k \xi^\beta e^{-i\xi(x)} e^{-P_p(y, \xi)} d\xi &= \int_{\xi \in \mathbb{V}^*} (P_p(y, \xi - i\nu))^k (\xi - i\nu)^\beta e^{-i(\xi - i\nu)(x)} e^{-P_p(y, \xi - i\nu)} d\xi \\ &= e^{-\nu(x)} \int_{\xi \in \mathbb{V}^*} (P_p(y, \xi - i\nu))^k (\xi - i\nu)^\beta e^{-i\xi(x)} e^{-P_p(y, \xi - i\nu)} d\xi. \end{aligned}$$

Thus, by virtue of the estimate (31),

$$\begin{aligned} \left| \int_{\mathbb{V}^*} (P_p(y, \xi))^k \xi^\beta e^{-i\xi(x)} e^{-P_p(y, \xi)} d\xi \right| &\leq C_\beta C_0^k k! e^{-\nu(x)} e^{M_2 R(\nu)} \int_{\mathbb{V}^*} e^{-M_1 R(\xi)} d\xi \\ &\leq C_\beta C_0^k k! e^{-(\nu(x) - M_2 R(\nu))} \end{aligned}$$

for all $x, y \in \mathbb{V}$ and $\nu \in \mathbb{V}^*$ where we have absorbed the integral of $\exp(-M_1 R(\xi))$ into C_β . Upon minimizing with respect to $\nu \in \mathbb{V}^*$, we have

$$\left| \int_{\mathbb{V}^*} (P_p(y, \xi))^k \xi^\beta e^{-i\xi(x)} e^{-P_p(y, \xi)} d\xi \right| \leq C_\beta C_0^k k! e^{-(M_2 R)^\#(x)} \leq C_\beta C_0^k k! e^{-MR^\#(x)} \quad (35)$$

for all x and $y \in \mathbb{V}$ because

$$-(M_2 R)^\#(x) = -\sup_{\nu} \{\nu(x) - M_2 R(\nu)\} = \inf_{\nu} \{-\nu(x) - M_2 R(\nu)\};$$

in this we see the natural appearance of the Legendre-Fenchel transform. The replacement of $(M_2 R)^\#(x)$ by $MR^\#(x)$ is done using Corollary 3.3 and, as required, the constant M is independent of k and β . Upon combining (30) and (35), we obtain the desired estimate (28). \square

As a simple corollary to the lemma, we obtain Proposition 2.11.

Proof of Proposition 2.11. Given a positive-homogeneous operator Λ , we invoke Proposition 2.5 to obtain \mathbf{v} and \mathbf{m} for which $\Lambda = \sum_{|\beta:\mathbf{m}|=2} a_\beta D_{\mathbf{v}}^\beta$. In other words, Λ is an $(2\mathbf{m}, \mathbf{v})$ -positive-semi-elliptic operator which consists only of its principal part. Consequently, the heat kernel K_Λ satisfies $K_\Lambda^t(x) = G_p(t, x; 0)$ for all $x \in \mathbb{V}$ and $t > 0$ and so we immediately obtain the estimate (11) from the lemma. \square

Making use of Hypothesis 5.2, a similar argument to that given in the proof of Lemma 5.3 yields the following lemma.

Lemma 5.4. *There is a positive constant M and, to each multi-index β , a positive constant C_β such that*

$$|D_{\mathbf{v}}^\beta[G_p(t, x; y+h) - G_p(t, x; y)]| \leq C_\beta t^{-(\mu_H + |\beta:2\mathbf{m}|)} |h|_{\mathbf{v}}^\alpha \exp(-tMR^\#(x/t))$$

for all $t > 0$, $x, y, h \in \mathbb{V}$. Here, in view of Hypothesis 5.2, α is the \mathbf{v} -Hölder continuity exponent for the coefficients of H .

Lemma 5.5. *Suppose that $g \in C_b((t_0, T] \times \mathbb{V})$ where $0 \leq t_0 < T < \infty$. Then, on any compact set $Q \subseteq (t_0, T] \times \mathbb{V}$,*

$$\int_{\mathbb{V}} G_p(t, x-y; y) g(s-t, y) dy \rightarrow g(s, x)$$

uniformly on Q as $t \rightarrow 0$. In particular, for any $f \in C_b(\mathbb{V})$,

$$\int_{\mathbb{V}} G_p(t, x-y; y) f(y) dy \rightarrow f(x)$$

uniformly on all compact subsets of \mathbb{V} as $t \rightarrow 0$.

Proof. Let Q be a compact subset of $(t_0, T] \times \mathbb{V}$ and write

$$\begin{aligned} & \int_{\mathbb{V}} G_p(t, x-y; y) g(s-t, y) dy \\ &= \int_{\mathbb{V}} G_p(t, x-y; x) g(s-t, y) dy + \int_{\mathbb{V}} [G_p(t, x-y; y) - G_p(t, x-y; x)] g(s-t, y) dy \\ &:= I_t^{(1)}(s, x) + I_t^{(2)}(s, x). \end{aligned}$$

Let $\epsilon > 0$ and, in view of Corollary 3.11, let K be a compact subset of \mathbb{V} for which

$$\int_{\mathbb{V} \setminus K} \exp(-MR^\#(z)) dz < \epsilon$$

where the constant M is that given in (28) of Lemma 5.3. Using the continuity of g , we have for sufficiently small $t > 0$,

$$\sup_{\substack{(s,x) \in Q \\ z \in K}} |g(s-t, x - t^E z) - g(s, x)| < \epsilon.$$

We note that, for any $t > 0$ and $x \in \mathbb{V}$,

$$\int_{\mathbb{V}} G_p(t, x - y; x) dy = e^{-tP_p(x, \xi)} \Big|_{\xi=0} = 1.$$

Appealing to Lemma 5.3 we have, for any $(s, x) \in Q$,

$$\begin{aligned} |I_t^{(1)}(s, x) - g(s, x)| &\leq \left| \int_{\mathbb{V}} G_p(t, x - y; x)(g(s - t, y) - g(s, x)) dy \right| \\ &\leq \int_{\mathbb{V}} |G_p(1, z; x)(g(s - t, x - t^E z) - g(s, x))| dz \\ &\leq 2\|g\|_{\infty} C \int_{\mathbb{V} \setminus K} \exp(-MR^{\#}(z)) dz \\ &\quad + C \int_K \exp(-MR^{\#}(z)) |(g(s - t, x - t^E z) - g(s, x))| dz \\ &\leq \epsilon C \left(2\|g\|_{\infty} + \|e^{-MR^{\#}}\|_1 \right); \end{aligned}$$

here we have made the change of variables: $y \mapsto t^E(x - y)$ and used the homogeneity of P_p to see that $t^{\mu_H} G_p(t, t^E z; x) = G_p(1, z; x)$. Therefore $I_t^{(1)}(s, x) \rightarrow g(s, x)$ uniformly on Q as $t \rightarrow 0$.

Let us now consider $I^{(2)}$. With the help of Lemmas 3.13 and 5.4 and by making similar arguments to those above we have

$$\begin{aligned} |I_t^{(2)}(s, x)| &\leq C\|g\|_{\infty} \int_{\mathbb{V}} t^{-\mu_H} |x - y|_{\mathbb{V}}^{\alpha} \exp(-MR^{\#}(t^{-E}(x - y))) dy \\ &\leq \|g\|_{\infty} C t^{\sigma} \int_{\mathbb{V}} t^{-\text{tr } E} (R^{\#}(t^{-E}(x - y)))^{\theta} \exp(-MR^{\#}(t^{-E}(x - y))) dy \\ &\leq \|g\|_{\infty} C t^{\sigma} \int_{\mathbb{V}} (R^{\#}(x))^{\theta} \exp(-MR^{\#}(z)) dz \leq \|g\|_{\infty} C' t^{\sigma} \end{aligned}$$

for all $s \in (t_0, T]$, $0 < t < s - t_0$ and $x \in \mathbb{V}$; here $0 < \sigma < 1$. Consequently, $I_t^{(2)}(s, x) \rightarrow 0$ uniformly on Q as $t \rightarrow 0$ and the lemma is proved. \square

Combining the results of Lemmas 5.3 and 5.5 yields at once:

Corollary 5.6. *For each $y \in \mathbb{V}$, $G_p(\cdot, \cdot - y; y)$ is a fundamental solution to (23).*

Step 2. Construction of ϕ and the integral equation

For $t > 0$ and $x, y \in \mathbb{V}$, put

$$\begin{aligned} K(t, x, y) &= -(\partial_t + H)G_p(t, x - y; y) \\ &= (H_p(y) - H)G_p(t, x - y; y) \\ &= \int_{\mathbb{V}^*} e^{-i\xi(x-y)} (P_p(y, \xi) - P(x, \xi)) e^{-tP_p(y, \xi)} d\xi \end{aligned}$$

and iteratively define

$$K_{n+1}(t, x, y) = \int_0^t \int_{\mathbb{V}} K_1(t-s, x, z) K_n(s, z, y) dz ds$$

where $K_1 = K$. In the sense of (27), note that $K_{n+1} = L^n K$.

We claim that for some $0 < \rho < 1$,

$$|K(t, x, y)| \leq C t^{-(\mu_H+1-\rho)} \exp(-MR^\#(t^{-E}(x-y))) \quad (36)$$

for all $x, y \in \mathbb{V}$ and $0 < t \leq T$ where M and C are positive constants. Indeed, observe that

$$|K(t, x, y)| \leq \sum_{|\beta:\mathbf{m}|=2} |a_\beta(y) - a_\beta(x)| |D_{\mathbb{V}}^\beta G_p(t, x-y; y)| + C \sum_{|\beta:\mathbf{m}|<2} |D_{\mathbb{V}}^\beta G_p(t, x-y; y)|$$

for all $x, y \in \mathbb{V}$ and $t > 0$ where we have used the fact that the coefficients of H are bounded. In view of Lemma 5.3, we have

$$\begin{aligned} |K(t, x, y)| \leq & \sum_{|\beta:\mathbf{m}|=2} |a_\beta(y) - a_\beta(x)| C t^{-(\mu_H+1)} \exp(-MR^\#(t^{-E}(x-y))) \\ & + C t^{-(\mu_H+\eta)} \exp(-MR^\#(t^{-E}(x-y))) \end{aligned}$$

for all $x, y \in \mathbb{V}$ and $0 < t \leq T$ where

$$\eta = \max\{|\beta : 2\mathbf{m}| : |\beta : \mathbf{m}| \neq 2 \text{ and } a_\beta \neq 0\} < 1.$$

Using Hypothesis 5.2, an appeal to Corollary 3.14 ensures that

$$\begin{aligned} |K(t, x, y)| \leq & C t^{\sigma-(\mu_H+1)} (R^\#(t^{-E}(x-y)))^\theta \exp(-MR^\#(t^{-E}(x-y))) \\ & + C t^{-(\mu_H+\eta)} \exp(-MR^\#(t^{-E}(x-y))) \end{aligned}$$

for all $x, y \in \mathbb{V}$ and $0 < t \leq T$ where θ is positive and $0 < \sigma < 1$. Our claim is then justified by choosing $\rho = \max\{\sigma, 1 - \eta\}$ and adjusting the constants C and M appropriately.

Taking cues from our heuristic discussion, we will soon form a series whose summands are the functions K_n for $n \geq 1$. In order to talk about the convergence of this series, our next task is to estimate these functions and in doing this we will observe two separate behaviors: a finite number of terms will exhibit singularities in t at the origin; the remainder of the terms will be absent of such singularities and will be estimated with the help of the Gamma function. We first address the terms with the singularities.

Lemma 5.7. *Let $0 < \rho < 1$ and $M > 0$ be as above. For any positive natural number n such that $\rho(n-1) \leq \mu_H + 1$ and $\epsilon > 0$ for which $\epsilon n < 1$, there is a constant $C_n(\epsilon) \geq 1$ such that*

$$|K_n(t, x, y)| \leq C_n(\epsilon) t^{-(\mu_H+1-n\rho)} \exp(-M(1-\epsilon n)R^\#(t^{-E}(x-y)))$$

for all $x, y \in \mathbb{V}$ and $0 < t \leq T$.

Proof. In view (36), it is clear that the estimate holds when $n = 1$. Let us assume the estimate holds for $n \geq 1$ such that $\rho n < 1 + \mu_H$ and $\epsilon > 0$ for which $\epsilon n < \epsilon(n + 1) < 1$. Then

$$|K_{n+1}(t, x, y)| \leq \int_0^t \int_{\mathbb{V}} C_n(\epsilon)(t-s)^{-(\mu_H+1-n\rho)} C_1(\epsilon) s^{-(\mu_H+1-\rho)} \\ \times \exp(-M_{\epsilon,n} R^\#((t-s)^{-E}(x-z))) \exp(-MR^\#(s^{-E}(z-y))) dz ds \quad (37)$$

for $x, y \in \mathbb{V}$ and $0 < t \leq T$ where we have set $M_{\epsilon,n} = M(1 - \epsilon n)$. Observe that

$$\begin{aligned} R^\#(t^{-E}(x-y)) &= \sup\{\xi(x-y) - tR(\xi)\} \\ &= \sup\{\xi(x-z) - (t-s)R(\xi) + \xi(z-y) - sR(\xi)\} \\ &\leq R^\#((t-s)^{-E}(x-z)) + R^\#(s^{-E}(z-y)) \end{aligned} \quad (38)$$

for all $x, y, z \in \mathbb{V}$ and $0 < s \leq t$ and therefore

$$\begin{aligned} &\exp(-M_{\epsilon,n} R^\#((t-s)^{-E}(x-z))) \exp(-MR^\#(s^{-E}(z-y))) \\ &\leq \exp(-M_{\epsilon,n+1} R^\#(t^{-E}(x-y))) \exp(-\epsilon n M (R^\#((t-s)^{-E}(x-z)) + R^\#(s^{-E}(z-y)))) \end{aligned} \quad (39)$$

Combining (37), (38) and (39) yields

$$\begin{aligned} &|K_{n+1}(t, x, y)| \\ &\leq C_1(\epsilon) C_n(\epsilon) \exp(-M_{\epsilon,n+1} R^\#(t^{-E}(x-y))) \int_0^t \int_{\mathbb{V}} (t-s)^{-(\mu_H+1-n\rho)} s^{-(\mu_H+1-\rho)} \\ &\quad \times \exp(-\epsilon n M (R^\#((t-s)^{-E}(x-z)) + R^\#(s^{-E}(z-y)))) dz ds \\ &\leq C_1(\epsilon) C_n(\epsilon) \exp(-M_{\epsilon,n+1} R^\#(t^{-E}(x-y))) \\ &\quad \times \left[(t/2)^{-(\mu_H+1-n\rho)} \int_0^{t/2} \int_{\mathbb{V}} s^{-(\mu_H+1-\rho)} \exp(-\epsilon n M R^\#(s^{-E}(z-y))) dz ds \right. \\ &\quad \left. + (t/2)^{-(\mu_H+1+\rho)} \int_{t/2}^t \int_{\mathbb{V}} (t-s)^{-(\mu_H+1-n\rho)} \exp(-\epsilon n M R^\#((t-s)^{-E}(x-z))) dz ds \right] \\ &\leq C_1(\epsilon) M_n(\epsilon) \exp(-M_{\epsilon,n+1} R^\#(t^{-E}(x-y))) \\ &\quad \times \left[(t/2)^{-(\mu_H+1-n\rho)} \int_0^{t/2} s^{-(1-\rho)} ds \int_{\mathbb{V}} \exp(-\epsilon n M R^\#(z)) dz \right. \\ &\quad \left. + (t/2)^{-(\mu_H+1+\rho)} \int_0^{t/2} s^{-(1-n)\rho} ds \int_{\mathbb{V}} \exp(-\epsilon n M R^\#(z)) dz \right] \\ &\leq C_{n+1}(\epsilon) t^{-(\mu_H+1-(n+1)\rho)} \exp(-M_{\epsilon,n+1} R^\#(t^{-E}(x-y))) \end{aligned}$$

for all $x, y \in \mathbb{V}$ and $t > 0$ where we have put

$$C_{n+1}(\epsilon) = C_1(\epsilon) C_n(\epsilon) \frac{n+1}{n\rho} 2^{\mu_H+(1-(n+1)\rho)} \int_{\mathbb{V}} \exp(-\epsilon n M R^\#(z)) dz$$

and made use of Corollary 3.11. □

Remark 6. The estimate (38) is an important one and will be used again. In the context of elliptic operators, i.e., where $R^\#(x) = C_m|x|^{2m/(2m-1)}$, the analogous result is captured in Lemma 5.1 of [35]. It is interesting to note that S. D. Eidelman worked somewhat harder to prove it. Perhaps this is because the appearance of the Legendre-Fenchel transform wasn't noticed.

It is clear from the previous lemma that for sufficiently large n , K_n is bounded by a positive power of t . The first such n is $\bar{n} := \lceil \rho^{-1}(\text{tr } E + 1) \rceil$. In view of the previous lemma,

$$|K_{\bar{n}}(t, x, y)| \leq C_{\bar{n}}(\epsilon) \exp(-M(1 - \epsilon\bar{n})R^\#(t^{-E}(x - y)))$$

for all $x, y \in \mathbb{V}$ and $0 < t \leq T$ where we have adjusted $C_{\bar{n}}(\epsilon)$ to account for this positive power of t . Let $\delta < 1/2$ and set

$$\epsilon = \frac{\delta}{\bar{n}}, \quad M_1 = M(1 - \delta) \quad \text{and} \quad C_0 = \max_{1 \leq n \leq \bar{n}} C_n(\epsilon).$$

Upon combining proceeding estimate with the estimates(36) and (38), we have

$$\begin{aligned} & |K_{\bar{n}+1}(t, x, y)| \\ & \leq C_0^2 \int_0^t \int_{\mathbb{V}} (t-s)^{-(\mu_H+(1-\rho))} \\ & \quad \times \exp(-MR^\#((t-s)^{-E}(x-z))) \exp(-M(1-\epsilon\bar{n})R^\#(s^{-E}(z-y))) ds dz \\ & \leq C_0^2 \exp(-M_1R^\#(t^{-E}(x-y))) \int_0^t \int_{\mathbb{V}} (t-s)^{-(\mu_H+(1-\rho))} \exp(-C\delta R^\#((t-s)^{-E}(z))) dz ds \\ & \leq C_0(C_0F) \frac{t^\rho}{\rho} \exp(-M_1R^\#(t^{-E}(x-y))) \end{aligned}$$

for all $x, y \in \mathbb{V}$ and $0 < t \leq T$ where

$$F = \int_{\mathbb{V}} \exp(-M\delta R^\#(z)) dz < \infty.$$

Let us take this a little further.

Lemma 5.8. *For every $k \in \mathbb{N}_+$,*

$$|K_{\bar{n}+k}(t, x, y)| \leq \frac{C_0}{\Gamma(\rho)} \frac{(C_0F\Gamma(\rho))^k}{k!} t^{\rho k} \exp(-M_1R^\#(t^{-E}(x-y))) \quad (40)$$

for all $x, y \in \mathbb{V}$ and $0 < t \leq T$. Here $\Gamma(\cdot)$ denotes the Gamma function.

Proof. The Euler-Beta function $B(\cdot, \cdot)$ satisfies the well-known identity $B(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$. Using this identity, one quickly obtains the estimate

$$\prod_{j=1}^{k-1} B(\rho, 1+j\rho) = \frac{\Gamma(\rho)^{k-1}}{\Gamma(1+k\rho)} \leq \frac{\Gamma(\rho)^{k-1}}{k!}.$$

It therefore suffices to prove that

$$|K_{\bar{n}+k}(t, x, y)| \leq C_0(C_0F)^k \prod_{j=0}^{k-1} B(\rho, 1 + j\rho) t^{k\rho} \exp(-M_1 R^\#(t^{-E}(x - y))) \quad (41)$$

for all $x, y \in \mathbb{V}$ and $0 < t \leq T$.

We first note that $B(\rho, 1) = \rho^{-1}$ and so, for $k = 1$, (41) follows directly from the calculation proceeding the lemma. We shall induct on k . By another application of (36) and (38), we have

$$\begin{aligned} J_{k+1}(t, x, y) &:= \left[C_0^2(C_0F)^k \prod_{j=0}^{k-1} B(\rho, 1 + j\rho) \right]^{-1} |K_{\bar{n}+k+1}(t, x, y)| \\ &\leq \int_0^t \int_{\mathbb{V}} (t-s)^{-(\mu_H+(1-\rho))} s^{-k\rho} \exp(-MR^\#((t-s)^{-E}(x-z))) \\ &\quad \times \exp(-M_1 R^\#(s^{-E}(z-y))) dz ds \\ &\leq \exp(-M_1 R^\#(t^{-E}(x-y))) \\ &\quad \times \int_0^t \int_{\mathbb{V}} (t-s)^{-(\mu_H+(1-\rho))} s^{-k\rho} \exp(-M\delta R^\#((t-s)^{-E}(x-z))) dz ds \end{aligned}$$

for all $x, y \in \mathbb{V}$ and $0 < t \leq T$. Upon making the changes of variables $z \rightarrow (t-s)^{-E}(x-z)$ followed by $s \rightarrow s/t$, we have

$$\begin{aligned} J_{k+1}(t, x, y) &\leq \exp(-M_1 R^\#(t^{-E}(x-y))) F \int_0^1 (t-st)^{\rho-1} (st)^{k\rho} t ds \\ &\leq \exp(-M_1 R^\#(t^{-E}(x-y))) F t^{(k+1)\rho} B(\rho, 1 + k\rho) \end{aligned}$$

for all $x, y \in \mathbb{V}$ and $0 < t \leq T$. Therefore (41) holds for $k+1$ as required. \square

Proposition 5.9. *Let $\phi : (0, T] \times \mathbb{V} \times \mathbb{V} \rightarrow \mathbb{C}$ be defined by*

$$\phi = \sum_{n=1}^{\infty} K_n.$$

This series converges uniformly for $x, y \in \mathbb{V}$ and $t_0 \leq t \leq T$ where t_0 is any positive constant. There exists $C \geq 1$ for which

$$|\phi(t, x, y)| \leq \frac{C}{t^{\mu_H+(1-\rho)}} \exp(-M_1 R^\#(t^{-E}(x-y))) \quad (42)$$

for all $x, y \in \mathbb{V}$ and $0 < t \leq T$ where M_1 and ρ are as in the previous lemmas. Moreover, the identity

$$\phi(t, x, y) = K(t, x, y) + \int_0^t \int_{\mathbb{V}} K(t-s, x, z) \phi(s, z, y) dz ds \quad (43)$$

holds for all $x, y \in \mathbb{V}$ and $0 < t \leq T$.

Proof. Using Lemmas 5.7 and 5.8 we see that

$$\sum_{k=1}^{\infty} |K_n(t, x, y)| \leq C_0 \left[\sum_{n=1}^{\bar{n}} t^{-(\mu_H + (1-n)\rho)} + \frac{1}{\Gamma(\rho)} \sum_{k=1}^{\infty} \frac{(C_0 F \Gamma(\rho))^k}{k!} t^{k\rho} \right] \exp(-M_1 R^\#(t^{-E}(x-y)))$$

for all $x, y \in \mathbb{V}$ and $0 < t \leq T$ from which (42) and our assertion concerning uniform convergence follow. A similar calculation and an application of Tonelli's theorem justify the following use of Fubini's theorem: For $x, y \in \mathbb{V}$ and $0 < t \leq T$,

$$\begin{aligned} \int_0^t \int_{\mathbb{V}} K(t-s, x, z) \phi(s, z, y) ds dz &= \sum_{n=1}^{\infty} \int_0^t \int_{\mathbb{V}} K(t-s, x, z) K_n(s, z, y) dz ds \\ &= \sum_{n=1}^{\infty} K_{n+1}(t, z, y) = \phi(t, x, y) - K(t, x, y) \end{aligned}$$

as desired. \square

The following Hölder continuity estimate for ϕ is obtained by first showing the analogous estimate for K and then deducing the desired result from the integral formula (43). As the proof is similar in character to those of the preceding two lemmas, we omit it. A full proof can be found in [37, p.80]. We also note here that the result is stronger than is required for our purposes (see its use in the proof of Lemma 5.12). All that is really required is that $\phi(\cdot, \cdot, y)$ satisfies the hypotheses (for f) in Lemma 5.11 for each $y \in \mathbb{V}$.

Lemma 5.10. *There exists $\alpha \in \mathbb{I}_+^d$ which is consistent with \mathbf{m} , $0 < \eta < 1$ and $C \geq 1$ such that*

$$|\phi(t, x+h, y) - \phi(t, x, y)| \leq \frac{C}{t^{\mu_H + (1-\eta)}} |h|_{\mathbb{V}}^\alpha \exp(-M_1 R^\#(t^{-E}(x-y)))$$

for all $x, y, h \in \mathbb{V}$ and $0 < t \leq T$.

Step 3. Verifying that Z is a fundamental solution to (20)

Lemma 5.11. *Let $\alpha \in \mathbb{I}_+^d$ be consistent with \mathbf{m} and, for $t_0 > 0$, let $f : [t_0, T] \times \mathbb{V} \rightarrow \mathbb{C}$ be bounded and continuous. Moreover, suppose that f is uniformly \mathbf{v} -Hölder continuous in x on $[t_0, T] \times \mathbb{V}$ of order α , by which we mean that there is a constant $C > 0$ such that*

$$\sup_{t \in [t_0, T]} |f(t, x) - f(t, y)| \leq C |x - y|_{\mathbb{V}}^\alpha$$

for all $x, y \in \mathbb{V}$. Then $u : [t_0, T] \times \mathbb{V} \rightarrow \mathbb{C}$ defined by

$$u(t, x) = \int_{t_0}^t \int_{\mathbb{V}} G_p(t-s, x-z; z) f(s, z) ds dz$$

is $(2\mathbf{m}, \mathbf{v})$ -regular on $(t_0, T) \times \mathbb{V}$. Moreover,

$$\partial_t u(t, x) = f(t, x) + \lim_{h \downarrow 0} \int_{t_0}^{t-h} \int_{\mathbb{V}} \partial_t G_p(t-s, x-z; z) f(s, z) dz ds \quad (44)$$

and for any β such that $|\beta : \mathbf{m}| \leq 2$, we have

$$D_{\mathbf{v}}^{\beta} u(t, x) = \lim_{h \downarrow 0} \int_{t_0}^{t-h} \int_{\mathbb{V}} D_{\mathbf{v}}^{\beta} G(t-s, x-z; z) f(s, z) dz ds \quad (45)$$

for $x \in \mathbb{V}$ and $t_0 < t < T$.

Before starting the proof, let us observe that, for each multi-index β ,

$$|D_{\mathbf{v}}^{\beta} G_p(t-s, x-z; z) f(s, z)| \leq C(t-s)^{-(\mu_H + |\beta:2\mathbf{m}|)} \exp(-MR^{\#}((t-s)^{-E}(x-z))) |f(s, z)|.$$

Using the assumption that f is bounded, we observe that

$$\begin{aligned} & \int_{t_0}^t \int_{\mathbb{V}} |D_{\mathbf{v}}^{\beta} G_p(t-s, x-z; z) f(s, z)| dz ds \\ & \leq C \int_{t_0}^t \int_{\mathbb{V}} (t-s)^{-\mu_H + |\beta:2\mathbf{m}|} \exp(-MR^{\#}((t-s)^{-E}(x-z))) dz ds \\ & \leq C \int_{t_0}^t \int_{\mathbb{V}} (t-s)^{-|\beta:2\mathbf{m}|} \exp(-MR^{\#}(z)) dz ds \\ & \leq C \int_{t_0}^t (t-s)^{-|\beta:2\mathbf{m}|} ds \end{aligned}$$

for all $t_0 \leq t \leq T$ and $x \in \mathbb{V}$. When $|\beta : \mathbf{m}| < 2$,

$$\int_{t_0}^t (t-s)^{-|\beta:2\mathbf{m}|} ds \quad (46)$$

converges and consequently

$$D_{\mathbf{v}}^{\beta} u(t, x) = \int_{t_0}^t \int_{\mathbb{V}} D_{\mathbf{v}}^{\beta} G_p(t-s, z-x; z) f(s, z) dz ds$$

for all $t_0 \leq t \leq T$ and $x \in \mathbb{V}$. From this it follows that $D_{\mathbf{v}}^{\beta} u(t, x)$ is continuous on $(t_0, T) \times \mathbb{V}$ and moreover (45) holds for such an β in view of Lebesgue's dominated convergence theorem. When $|\beta : \mathbf{m}| = 2$, (46) does not converge and hence the above argument fails. The main issue in the proof below centers around using \mathbf{v} -Hölder continuity to remove this obstacle.

Proof. Our argument proceeds in two steps. The first step deals with the spatial derivatives of u . Therein, we prove the asserted x -regularity and show that the formula (45) holds. In fact, we only need to consider the case where $|\beta : \mathbf{m}| = 2$; the case where $|\beta : \mathbf{m}| < 2$ was already treated in the paragraph preceding the proof. In the second step, we address the time derivative of u . As we will see, (44) and the asserted t -regularity are partial consequences of the results proved in Step 1; this is, in part, due to the fact that the time derivative of G_p can be exchanged for spatial derivatives. The regularity shown in the two steps together will automatically ensure that u is $(2\mathbf{m}, \mathbf{v})$ -regular on $(t_0, T) \times \mathbb{V}$.

Step 1. Let β be such that $|\beta : \mathbf{m}| = 2$. For $h > 0$ write

$$u_h(t, x) = \int_{t_0}^{t-h} \int_{\mathbb{V}} G_p(t-s, x-z; z) f(s, z) dz ds$$

and observe that

$$D_{\mathbb{V}}^{\beta} u_h(t, x) = \int_{t_0}^{t-h} \int_{\mathbb{V}} D_{\mathbb{V}}^{\beta} G_p(t-s, x-z; z) f(s, z) dz ds$$

for all $t_0 \leq t-h < t \leq T$ and $x \in \mathbb{V}$; it is clear that $D_{\mathbb{V}}^{\beta} u_h(t, x)$ is continuous in t and x . The fact that we can differentiate under the integral sign is justified because t has been replaced by $t-h$ and hence the singularity in (46) is avoided in the upper limit. We will show that $D_{\mathbb{V}}^{\beta} u_h(t, x)$ converges uniformly on all compact subsets of $(t_0, T) \times \mathbb{V}$ as $h \rightarrow 0$. This, of course, guarantees that $D_{\mathbb{V}}^{\beta} u(x, t)$ exists, satisfies (45) and is continuous on $(t_0, T) \times \mathbb{V}$. To this end, let us write

$$\begin{aligned} D_{\mathbb{V}}^{\beta} u_h(t, x) &= \int_{t_0}^{t-h} \int_{\mathbb{V}} D_{\mathbb{V}}^{\beta} G_p(t-s, x-z; z) (f(s, z) - f(s, x)) dz ds \\ &\quad + \int_{t_0}^{t-h} \int_{\mathbb{V}} D_{\mathbb{V}}^{\beta} G_p(t-s, x-z; z) f(s, x) dz ds \\ &=: I_h^{(1)}(t, x) + I_h^{(2)}(t, x). \end{aligned}$$

Using our hypotheses, Corollary 3.8 and Lemma 3.13, for some $0 < \sigma < 1$ and $\theta > 0$, there is $M > 0$ such that

$$|f(s, z) - f(s, x)| \leq C(t-s)^{\sigma} (R^{\#}((t-s)^{-E}(x-z)))^{\theta}$$

for all $x, z \in \mathbb{V}$, $t \in [t_0, T]$ and $s \in [t_0, t]$; consequently

$$\begin{aligned} |D_{\mathbb{V}}^{\beta} G_p(t-s, x-z; z) (f(s, z) - f(s, x))| &\leq C(t-s)^{-(\mu_H+1)} t^{\sigma} (R^{\#}(t^{-E}(x-z)))^{\theta} \exp(-MR^{\#}((t-s)^{-E}(x-z))) \\ &\leq C(t-s)^{-(\mu_H+(1-\sigma))} \exp(-MR^{\#}(t-s)^{-E}(x-z)) \end{aligned}$$

for all $x, z \in \mathbb{V}$, $t \in [t_0, T]$ and $s \in [t_0, t]$. This estimate guarantees that

$$I^{(1)}(t, x) := \int_{t_0}^t \int_{\mathbb{V}} D_{\mathbb{V}}^{\beta} G_p(t-s, x-z; z) (f(s, z) - f(s, x)) dz ds$$

exists for each $t \in [t_0, T]$ and $x \in \mathbb{V}$. Moreover, for all $t_0 \leq t-h < t \leq T$ and $x \in \mathbb{V}$,

$$\begin{aligned} |I^{(1)}(t, x) - I_h^{(1)}(t, x)| &\leq \int_{t-h}^t \int_{\mathbb{V}} |D_{\mathbb{V}}^{\beta} G_p(t-s, x-z; z) (f(s, z) - f(s, x))| dz ds \\ &\leq C \int_{t-h}^t \int_{\mathbb{V}} (t-s)^{\sigma-1} \exp(-MR^{\#}(z)) dz ds \leq Ch^{\sigma}. \end{aligned}$$

From this we see that $\lim_{h \downarrow 0} I_h^{(1)}(t, x)$ converges uniformly on all compact subsets of $(t_0, T) \times \mathbb{V}$.

We claim that for some $0 < \rho < 1$, there exists $C > 0$ such that

$$\left| \int_{\mathbb{V}} D_{\mathbb{V}}^{\beta} G_p(t-s, x-z; z) dz \right| \leq C(t-s)^{-(1-\rho)} \quad (47)$$

for all $x \in \mathbb{V}$ and $s \in [t_0, t]$. Indeed,

$$\begin{aligned} & \int_{\mathbb{V}} D_{\mathbb{V}}^{\beta} G_p(t-s, x-z; z) dz \\ &= \int_{\mathbb{V}} D_{\mathbb{V}}^{\beta} [G_p(t-s, x-z; z) - G_p(t-s, x-z; y)] \Big|_{y=x} dz + [D_{\mathbb{V}}^{\beta} \int_{\mathbb{V}} G_p(t-s, x-z; y) dz] \Big|_{y=x}. \end{aligned}$$

The first term above is estimate with the help of Lemma 5.4 and by making arguments analogous to those in the previous paragraph; the appearance of ρ follows from an obvious application of Lemma 3.13. This term is bounded by $C(t-s)^{-(1-\rho)}$. The second term is clearly zero and so our claim is justified.

By essentially repeating the arguments made for $I_h^{(1)}$ and making use of (47), we see that

$$\lim_{h \downarrow 0} I_h^{(2)}(t, x) = I^{(2)}(t, x) =: \int_{t_0}^t \int_{\mathbb{V}} D_{\mathbb{V}}^{\beta} G_p(t-s, x-z; z) f(s, x) dz ds$$

where this limit converges uniformly on all compact subsets of $(t_0, T) \times \mathbb{V}$.

Step 2. It follows from Leibnitz' rule that

$$\begin{aligned} \partial_t u_h(x, t) &= \int_{\mathbb{V}} G_p(h, x-z; z) f(t-h, z) dz + \int_{t_0}^{t-h} \int_{\mathbb{V}} \partial_t G_p(t-s, x-z; z) f(s, z) dz ds \\ &=: J_h^{(1)}(t, x) + J_h^{(2)}(t, x) \end{aligned}$$

for all $t_0 < t-h < t < T$ and $x \in \mathbb{V}$. Now, in view of Lemma 5.5 and our hypotheses concerning f ,

$$\lim_{h \downarrow 0} J_h^{(1)}(t, x) = f(t, x)$$

where this limit converges uniformly on all compact subsets of $(t_0, T) \times \mathbb{V}$.

Using the fact that $\partial_t G_p(t-s, x-z; z) = -H_p(z)G_p(t-s, x-z; z)$, we see that

$$\begin{aligned} \lim_{h \downarrow 0} J_h^{(2)}(t, x) &= \lim_{h \downarrow 0} \int_0^{t-h} \int_{\mathbb{V}} \left(- \sum_{|\beta: \mathbf{m}|=2} a_{\beta}(z) D_{\mathbb{V}}^{\beta} \right) G_p(t-s, x-z; z) f(s, z) dz ds \\ &= - \sum_{|\beta: \mathbf{m}|=2} \lim_{h \downarrow 0} \int_0^{t-h} \int_{\mathbb{V}} D_{\mathbb{V}}^{\beta} G_p(t-s, x-z; z) (a_{\beta}(z) f(s, z)) dz ds \end{aligned}$$

for all $t \in (t_0, T)$ and $x \in \mathbb{V}$. Because the coefficients of H are \mathbf{v} -Hölder continuous and bounded, for each β , $a_{\beta}(z) f(s, z)$ satisfies the same condition we have required for f and so, in view of Step 1, it follows that $J_h^{(2)}(t, x)$ converges uniformly on all compact subsets of $(t_0, T) \times \mathbb{V}$ as $h \rightarrow 0$. We thus conclude that $\partial_t u(t, x)$ exists, is continuous on $(t_0, T) \times \mathbb{V}$ and satisfies (44). \square

Lemma 5.12. Let $W : (0, T] \times \mathbb{V} \times \mathbb{V} \rightarrow \mathbb{C}$ be defined by

$$W(t, x, y) = \int_0^t \int_{\mathbb{V}} G_p(t-s, x-z; z) \phi(s, z, y) dz ds,$$

for $x, y \in \mathbb{V}$ and $0 < t \leq T$. Then, for each $y \in \mathbb{V}$, $W(\cdot, \cdot, y)$ is $(2\mathbf{m}, \mathbf{v})$ -regular on $(0, T) \times \mathbb{V}$ and satisfies

$$(\partial_t + H)W(t, x, y) = K(t, x, y). \quad (48)$$

for all $x, y \in \mathbb{V}$ and $t \in (0, T)$. Moreover, there are positive constants C and M for which

$$|W(t, x, y)| \leq Ct^{-\mu_H + \rho} \exp(-MR^\#(t^{-E}(x-y))) \quad (49)$$

for all $x, y \in \mathbb{V}$ and $0 < t \leq T$ where ρ is that which appears in Lemma 5.7.

Proof. The estimate (49) follows from (28) and (42) by an analogous computation to that done in the proof of Lemma 5.7. It remains to show that, for each $y \in \mathbb{V}$, $W(\cdot, \cdot, y)$ is $(2\mathbf{m}, \mathbf{v})$ -regular and satisfies (48) on $(0, T) \times \mathbb{V}$. These are both local properties and, as such, it suffices to examine them on $(t_0, T) \times \mathbb{V}$ for an arbitrary but fixed $t_0 > 0$. Let us write

$$\begin{aligned} W(t, x, y) &= \int_{t_0}^t \int_{\mathbb{V}} G_p(t-s, x-z; z) \phi(s, z, y) dz ds + \int_0^{t_0} \int_{\mathbb{V}} G_p(t-s, x-z; z) \phi(s, z, y) dz ds \\ &=: W_1(t, x, y) + W_2(t, x, y) \end{aligned}$$

for $x, y \in \mathbb{V}$ and $t_0 < t < T$. In view of Lemmas 5.10 and 5.11, for each $y \in \mathbb{V}$, $W_1(\cdot, \cdot, y)$ is $(2\mathbf{m}, \mathbf{v})$ -regular on $(t_0, T) \times \mathbb{V}$ and

$$\begin{aligned} (\partial_t + H)W_1(t, x, y) &= \partial_t W_1(t, x, y) + \sum_{|\beta: \mathbf{m}| \leq 2} a_\beta(x) D_{\mathbb{V}}^\beta W_1(t, x, y) \\ &= \phi(t, x, y) + \lim_{h \downarrow 0} \int_{t_0}^{t-h} \int_{\mathbb{V}} \partial_t G_p(t-s, x-z; z) \phi(s, z, y) dz dy \\ &\quad + \lim_{h \downarrow 0} \int_{t_0}^{t-h} \int_{\mathbb{V}} \sum_{|\beta: \mathbf{m}| \leq 2} a_\beta(x) D_{\mathbb{V}}^\beta G_p(t-s, x-z; z) \phi(s, z, y) dz ds \\ &= \phi(t, x, y) + \lim_{h \downarrow 0} \int_{t_0}^{t-h} \int_{\mathbb{V}} (\partial_t + H)G_p(t-s, x-z; z) \phi(s, z, y) dz ds \\ &= \phi(t, x, y) - \lim_{h \downarrow 0} \int_{t_0}^{t-h} \int_{\mathbb{V}} K(t-s, x, z) \phi(s, z, y) dz ds \end{aligned} \quad (50)$$

for all $x \in \mathbb{V}$ and $t_0 < t < T$; here we have used the fact that

$$(\partial_t + H)G_p(t-s, x-z; z) = -K(t-s, x, z).$$

Treating W_2 is easier because $G_p(t-s, x-z, z)$ and its derivatives remain bounded for $z, x \in \mathbb{V}$ and $0 < s \leq t_0$. Consequently, derivatives may be taken under the integral sign and so it follows that, for each $y \in \mathbb{V}$, $W_2(\cdot, \cdot, y)$ is $(2\mathbf{m}, \mathbf{v})$ -regular on $(t_0, T) \times \mathbb{V}$ and

$$(\partial_t + H)W_2(t, x, y) = - \int_0^{t_0} \int_{\mathbb{V}} K(t-s, x, z) \phi(s, z, y) dz ds \quad (51)$$

for $x \in \mathbb{V}$ and $t_0 < t < T$. We can thus conclude that, for each $y \in \mathbb{V}$, $W(\cdot, \cdot, y)$ is $(2\mathbf{m}, \mathbf{v})$ -regular on $(t_0, T) \times \mathbb{V}$ and, by combining (50) and (51),

$$(\partial_t + H)W(t, x, y) = \phi(t, x, y) - \lim_{h \downarrow 0} \int_0^{t-h} \int_{\mathbb{V}} K(t-s, x, z) \phi(s, z, y) dz ds$$

for $x \in \mathbb{V}$ and $t_0 < t < T$. By (36), Proposition 5.9 and the Dominated Convergence Theorem,

$$\begin{aligned} \lim_{h \downarrow 0} \int_0^{t-h} \int_{\mathbb{V}} K(t-s, x, z) \phi(s, z, y) dz ds &= \int_0^t \int_{\mathbb{V}} K(t-s, x, z) \phi(s, z, y) dz ds \\ &= \phi(t, x, y) - K(t, x, y) \end{aligned}$$

and therefore

$$(\partial_t + H)W(t, x, y) = K(t, x, y)$$

for all $x, y \in \mathbb{V}$ and $t_0 < t < T$. □

The theorem below is our main result. It is a more refined version of Theorem 5.1 because it gives an explicit formula for the fundamental solution Z ; in particular Theorem 5.1 is an immediate consequence of the result below.

Theorem 5.13. *Let H be a uniformly $(2\mathbf{m}, \mathbf{v})$ -positive-semi-elliptic operator. If H satisfies Hypothesis 5.2 then $Z : (0, T] \times \mathbb{V} \times \mathbb{V} \rightarrow \mathbb{C}$, defined by*

$$Z(t, x, y) = G_p(t, x - y; y) + W(t, x, y) \tag{52}$$

for $x, y \in \mathbb{V}$ and $0 < t \leq T$, is a fundamental solution to (20). Moreover, there are positive constants C and M for which

$$|Z(t, x, y)| \leq \frac{C}{t^{\mu_H}} \exp\left(-tMR^\# \left(\frac{x-y}{t}\right)\right) \tag{53}$$

for all $x, y \in \mathbb{V}$ and $0 < t \leq T$.

Proof. As $0 < \rho < 1$, (49) and Lemma 5.3 imply the estimate (53). In view of Lemma 5.12 and Corollary 5.6, for each $y \in \mathbb{V}$, $Z(\cdot, \cdot, y)$ is $(2\mathbf{m}, \mathbf{v})$ -regular on $(0, T) \times \mathbb{V}$ and

$$\begin{aligned} (\partial_t + H)Z(t, x, y) &= (\partial_t + H)G_p(t, x - y, y) + (\partial_t + H)W(t, x, y) \\ &= -K(t, x, y) + K(t, x, y) = 0 \end{aligned}$$

for all $x \in \mathbb{V}$ and $0 < t < T$. It remains to show that for any $f \in C_b(\mathbb{V})$,

$$\lim_{t \rightarrow 0} \int_{\mathbb{V}} Z(t, x, y) f(y) dy = f(x)$$

for all $x \in \mathbb{V}$. Indeed, let $f \in C_b(\mathbb{V})$ and, in view of (49), observe that

$$\begin{aligned} \left| \int_{\mathbb{V}} W(t, x, y) f(y) dy \right| &\leq Ct^\rho \|f\|_\infty \int_{\mathbb{V}} t^{-\mu_H} \exp(-MR^\#(t^{-E}(x-y))) dy \\ &\leq Ct^\rho \|f\|_\infty \int_{\mathbb{V}} \exp(-MR^\#(y)) dy \leq Ct^\rho \|f\|_\infty \end{aligned}$$

for all $x \in \mathbb{V}$ and $0 < t \leq T$. An appeal to Lemma 5.5 gives, for each $x \in \mathbb{V}$,

$$\begin{aligned} \lim_{t \rightarrow 0} \int_{\mathbb{V}} Z(t, x, y) f(y) dy &= \lim_{t \rightarrow 0} \int_{\mathbb{V}} G_p(t, x - y; y) f(y) dy + \lim_{t \rightarrow 0} \int_{\mathbb{V}} W(t, x, y) f(y) dy \\ &= f(x) + 0 = f(x) \end{aligned}$$

as required. In fact, the above argument guarantees that this convergence happens uniformly on all compact subsets of \mathbb{V} . \square

We remind the reader that implicit in the definition of fundamental solution to (20) is the condition that Z is $(2\mathbf{m}, \mathbf{v})$ -regular. In fact, one can further deduce estimates for the spatial derivatives of Z , $D_{\mathbf{v}}^{\beta} Z$, of the form (11) for all β such that $|\beta : 2\mathbf{m}| \leq 1$ (see [37, p. 92]). Using the fact that Z satisfies (20) and H 's coefficients are bounded, an analogous estimate is obtained for a single t derivative of Z .

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