Math 439: Homework 6

Please complete the following exercises below and write up your solutions consistent with the directions in the syllabus. Your solutions are due on Wednesday, November 6th at 11:00AM in the appropriate box outside my office door (please drop it in the box on your way to class). If you get stuck on any part of the homework, please come and see me. More importantly, have fun!

Exercise 1. Please do Exercises 7.A, 7.B, and 7.C in Bartle's book.

Exercise 2. Please look at, convince yourself you can do, but do not do Exercises 7.N and 7.O in Bartle's book.

Exercise 3. Please do Exercises 8.N, 8.O, and 8.P in Bartle's book.

Exercise 4. Please do Exercise 8.V in Bartle's book. The $p = 2$ version of the Riesz representation he is referring to is the following:

Theorem A. *Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$. If G is a continuous linear functional on H, i.e., $G : H \to \mathbb{C}$ is a linear function for which there is some constant M for which*

$$|G(x)| \leq M\|x\|$$

where $\|x\| = \sqrt{\langle x, x \rangle}$, then there is a unique element $y \in H$ for which

$$G(x) = \langle x, y \rangle$$

for all $x \in H$. Furthermore,

$$\|y\| = \inf\{M \in \mathbb{R} : |G(x)| \leq M\|x\| \ \forall x \in H\} = \sup\{|G(x)| : x \in H \text{ and } \|x\| = 1\}.$$

This version of Riesz's theorem for Hilbert spaces says, essentially, that every continuous linear functional is simply an inner product against some fixed element of the space. Okay, now how Bartle wants you to apply the theorem is in the setting of a measure space. So here is a fact you can use:

Theorem B. *Let $(X, \mathbb{X}, \nu)$ be a measure space. Then $L^2(X, \mathbb{X}, \nu)$, the set of (equivalence classes of) measurable functions $f : X \to \mathbb{C}$ for which*

$$\|f\|_2 = \left(\int_X |f(x)|^2 \, d\mu(x) \right)^{1/2} < \infty,$$

is a Hilbert space with inner product

$$\langle f, g \rangle = \int_X f(x)\overline{g(x)} \, d\nu$$

and the corresponding induced norm is simply the L^2 norm, i.e., $\|f\| = \sqrt{\langle f, f \rangle} = \|f\|_2$.

Okay, I think everything for Bartle is real valued (so his elements of L^2 are real valued (and the conjugation in the inner product is unnecessary) and his linear functionals take values in $\mathbb{R}$. So, feel free to adjust the above to that – but I wanted to state it over $\mathbb{C}$ so you know about it.

Exercise 5. Please do Bartle's exercises, 9.G, 9.H, and 9.I (this shows that Lebesgue measure is a so-called Radon measure).

Exercise 6. Please do Bartle's Exercises 9.S, 9.T, and 9.U.

Exercise 7. Please do Bartle's Exercises 10.G, and 10.P.

Exercise 8. Please do Bartle's Exercises 10.J, 10.N, and 10.O.