Math 439: Homework 5

Please complete the following exercises below and write up your solutions consistent with the directions in the syllabus. Your solutions are due on Wednesday+1, October 23th+1 at 11:00AM in the appropriate box outside my office door (please drop it in the box on your way to class). If you get stuck on any part of the homework, please come and see me. More importantly, have fun!

Exercise 1. Go outside and look at the fall foliage. Perhaps, take a walk on Runnals Hill and think about some of these exercises (in the spirit of G. Polýa).

Exercise 2. Please do Exercises 5.I and 5.J of Bartle.

Exercise 3. Please do Exercises 5.K, 5.L, and 5.M of Bartle.

Exercise 4. Please do Exercises 5.N and 5.O of Bartle. See Hint for 5.0 at end.

Exercise 5. Please do Exercises 6.D, 6.E, and 6.F of Bartle.

Exercise 6. This is a more well-developed version of Exercise 6.G. Please do the following:

1. Prove:

   Proposition A. *Consider the measure space* $(\mathbb{N}, 2^{\mathbb{N}}, \#)$ *where* $\#$ *is counting measure and, for* $1 \le p \le \infty$, *define* $\ell^p(\mathbb{N}) := L_p(\mathbb{N}, 2^{\mathbb{N}}, \#)$. *Then*
   $$\ell^p(\mathbb{N}) \subseteq \ell^q(\mathbb{N})$$
   whenever $1 \le p \le q \le \infty$. *Further, show this set containment is proper when* $p < q$.

   Hint: Holder works. Also, you could argue that $n \mapsto a_n^{q-p}$ *is bounded and go from there.*

2. Prove:

   Proposition B. *Let* $(X, \mathbb{X}, \mu)$ *be a finite measure space, i.e.,* $\mu(X) < \infty$, *and set* $L^p(X) = L_p(X, \mathbb{X}, \mu)$. *Then*
   $$L^p(X) \subseteq L^q(X)$$
   whenever $1 \le q \le p \le \infty$.

   Hint: Note that the constant function $g = 1 \in L^r(X)$ *for all* r. *Then use Hölder.*

3. Let's now consider a measure space that is distinctly different from the previous measure spaces: $(\mathbb{R}, \mathcal{L}_1, \lambda_1)$, where $\mathcal{L}_1$ is the Lebesgue σ-algebra and λ_1 is Lebesgue measure. Note that this measure space gives 0 measure to all singletons (so it isn't like $\mathbb{N}$ with counting measure) and it certainly isn't a finite measure space (so it isn't like the second). Set $L^p(\mathbb{R}) = L_p(\mathbb{R}, \mathcal{L}_1, \lambda_1)$. In this case, show that
   $$L^p(\mathbb{R}) \not\subseteq L^q(\mathbb{R})$$
   for any $p, q \in [1, \infty]$ unless $p = q$.

Hint: First, try using nice rational functions restricted to $(0, 1]$, *e.g.,* $1/x^s \times \chi_{(0,\infty)}(x)$ *for various values of* s. *You can use the fact that Riemann and Lebesgue integrals coincide for such functions – so you know how to compute things. Now try other rational functions restricted to other intervals...*

Finally, here is a hint for 5.0: Suppose that, for $f_n \in L^1$, we have
$$\sum_{n=1}^{\infty} \|f_n\|_1 < \infty.$$

We want to use LDCT to show that

$$\int \sum_{n=1}^{\infty} f_n(x)\, d\mu = \sum_{n=1}^{\infty} \int f_n(x)\, d\mu$$

however, we aren't guaranteed, *a priori*, that the series

$$\sum_{n=1}^{\infty} f_n(x)$$

converges at all. So this is our task.

Lemma C. *Under the hypotheses above, the series*

$$\sum_{n=1}^{\infty} f_n(x)$$

is absolutely convergent μ-almost everywhere.

Proof. For each $n \geq 1$, set

$$s_n(x) = \sum_{k=1}^{n} |f_n(x)|$$

for $x \in X$. To show that the desired series is absolutely convergent μ-almost everywhere, it is enough to show that its sequence of partial sums $(s_n(x))$ is Cauchy μ-almost everywhere. Our strategy will be to show that the set on which this sequence is not Cauchy has measure zero.

To this end, let M and N be positive natural numbers and set

$$E_N^M = \left\{ x \in X : |s_n(x) - s_m(x)| \geq \frac{1}{M} \;\text{ for some }\; n, m \geq N \right\}.$$

For fixed M and N, observe that Observe that, for any $x \in E_N^M$,

$$1 \leq M|s_n(x) - s_m(x)| = M \sum_{k=N}^{\max\{n,m\}} |f_k(x)| \leq M \sum_{k=N}^{\infty} |f_k(x)|$$

and therefore[1]

$$\mu\left(E_N^M\right) = \int_{E_N} 1\, d\mu \leq \int_{E_N} M \sum_{k=N}^{\infty} |f_k(x)|\, d\mu \leq M \sum_{k=N}^{\infty} \int_{E_N} |f_k(x)|\, d\mu = M \sum_{k=N}^{\infty} \|f_k\|_1.$$

Let's now fix M. As we have assumed that the series of L^1 norms is convergent, this gives us that

$$\lim_{N\to\infty} \mu\left(E_N^M\right) = M \lim_{N\to\infty} \sum_{k=N}^{\infty} \|f_k\|_1 = 0.$$

Further, it is not difficult to show that, for fixed M, the sets E_N^M are nested and so

$$\mu\left(\bigcap_{N=1}^{\infty} E_N^M\right) = \lim_{N\to\infty} \mu(E_N) = 0$$

by continuity of measure. In other words, we have proved that: For each $M \in \mathbb{N}$, the set

$$B^M = \bigcap_{N=1}^{\infty} E_N^M$$

[1] For those probability fans out there, the following is a version of Markov's inequality.

has

$$\mu(B^M) = 0.$$

So, if we set

$$B = \bigcup_{M \in \mathbb{N}} B^M$$

(Of course B stands for *bad*), we have

$$\mu(B) \leq \sum_{M=1}^{\infty} \mu(B^M) = \sum 0 = 0$$

by the "union bound". Now, let $x \in G = X/B$. We claim that the sequence $(s_n(x))$ is Cauchy. To see this, let $\epsilon > 0$ and select M such that $\frac{1}{M} < \epsilon$. Since $x \notin B$,

$$x \notin B^M = \bigcap_{N=1}^{\infty} F_N^M$$

so that there is some $N \in \mathbb{N}$ for which

$$x \in \left\{ x \in X : |s_n(x) - s_m(x)| \geq \frac{1}{M} \quad \text{for some} \quad n, m \geq N \right\}^c$$

In other words, for all $n, m \geq N$,

$$|s_n(x) - s_m(x)| < \frac{1}{M} < \epsilon.$$

We have therefore shown that, for each $x \in G = X \setminus B$, $(s_n(x))$ is Cauchy and therefore convergent. Thus, we may conclude that the series

$$\sum_{k=1}^{\infty} f_k(x)$$

is absolutely convergent for all $x \in G = X \setminus B$. Since $\mu(B) = 0$, this means that the series converges μ-almost everywhere. □