# Math 338: Homework 5

Please complete the following exercises below and write up your solutions consistent with the directions in the syllabus. Your solutions are due on Thursday, March 12th at 10:00AM in the appropriate box outside my office door. If you get stuck on any part of the homework, please come and see me. More importantly, have fun!

For the first part of the homework, we will focus on sequences and series of real and complex numbers.

**Exercise 1.** Let $\{s_n\}$ be a sequence of real numbers. For each $n \in \mathbb{N}_+$, define

$$x_n = \frac{s_1 + s_2 + \cdots + s_n}{n}.$$

In this way, we form a new sequence $\{x_n\}_{n\in\mathbb{N}}$ called the *Cesáro* means of $\{s_n\}$.

1. Prove the following: If $\{s_n\}$ converges to $s$, then $\{x_n\}$ converges to $s$.
2. Given an example of a sequence which doesn't converge but its Cesáro means converge. Justify your answer.

**Exercise 2** (All things Cauchy)**.** We recall, a sequence of complex numbers $\{s_n\}$ is said to be *Cauchy* if, for every $\epsilon > 0$, there exists $N \in \mathbb{N}_+$, such that

$$|s_m - s_n| < \epsilon \tag{1}$$

whenever $n, m \geq N$. It is not hard to see that the *Cauchy condition* (1) is equivalent to the following condition: For all $\epsilon > 0$, there exits $N \in \mathbb{N}_+$, such that

$$|s_{n+k} - s_n| < \epsilon \tag{2}$$

whenever $n \geq N$ and $k \geq 0$. (You should, at least, convince yourself that these are equivalent).

1. Show directly that $\{\frac{n+2}{n}\}_{n\in\mathbb{N}_+}$ is a Cauchy sequence of real numbers, i.e., prove that it satisfies (1) (or (2)).
2. Consider the sequence $\{\sqrt{n}\}_{n\in\mathbb{N}_+}$.
   (a) Show that this sequence is not convergent and therefore not Cauchy.
   (b) Show that, for any $k \geq 0$,
   $$\lim_{n\to\infty}\left(\sqrt{n+k} - \sqrt{n}\right) = 0.$$
   (c) Pertaining to the sequence $\{\sqrt{n}\}$, how does the limit established in the previous item differ from the condition (2)?

Now, consider the following definition

**Definition A.** A sequence $\{s_n\}$ of complex number is said to be **Super-Duper-Cauchy** if

$$|s_{n+1} - s_n| \leq 2^{-n} \tag{3}$$

for all $n \in \mathbb{N}_+$.

3. Give an example of a Super-Duper-Cauchy sequence. Give an example of a Cauchy sequence which is not Super-Duper-Cauchy.
4. With the help of the triangle inequality, show that if a sequence $\{s_n\}$ satisfies (3), then
   $$|s_m - s_n| \leq \sum_{k=n}^{m-1} 2^{-k}$$
   whenever $n, m$ are natural numbers such that $n < m$.

5. Use the estimate established in the previous item to prove the following proposition:

   **Proposition B.** *Let $\{s_n\}$ be a Super-Duper-Cauchy sequence. Then $\{s_n\}$ is Cauchy (and therefore convergent).*

   **Hint:** The proof of Theorem 3.26 in the textbook might be helpful.

**Exercise 3.** Please do Exercise 7 of Chapter 3 in Rudin. Hint: Cauchy-Schwarz will help your analysis of partial sums.

**Exercise 4.** Please do Exercise 9 in Chapter 3 in Rudin.

**Exercise 5.** (Abel's Test) Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers. Prove the following statements

1. If $\{a_n\}$ is a bounded sequence and the series $\sum_n b_n$ is absolutely convergent, then the series $\sum_n a_n b_n$ is absolutely convergent (and therefore converges). **Hint:** Use the Cauchy criterion, Theorem 3.22.
2. If $\{a_n\}$ is a non-increasing sequence of non-negative numbers and $\sum b_n$ converges, then $\sum_n a_n b_n$ converges. **Hint:** By assumption, $a_1 \geq a_2 \geq a_3 \geq \cdots 0$. Abbott's exercises 2.7.12 and 2.7.14 outline a method that works.
3. Comment on whether or not these results hold if $\{a_n\}$ or $\{b_n\}$ is taken to be complex.

For the final part of the homework, we shall consider the Hausdorff metric on the space of shapes. Recall, for a non-empty compact set (think a $d$-cell) $K \subseteq \mathbb{R}^d$, we consider the following collection of sets

$$\mathcal{S}(K) := \{A \subseteq K : A \text{ is non-empty compact.}\}.$$

We shall call every element $A \in \mathcal{S}(K)$ a *shape* and $\mathcal{S}(K)$ the *space of shapes*. Given two shapes $A, B \in \mathcal{S}(K)$, we define

$$d_H(A, B) = \inf\{\delta \geq 0 : A \subseteq N_\delta(B) \text{ and } B \subseteq N_\delta(A)\};$$

here, $N_\delta(A) = \{x \in \mathbb{R}^d : |x - y| < \delta \text{ for some } y \in A\}$. You can think of $N_\delta(A)$ as a slightly enlarged/ballooned extension of the set $A$. In class, we proved:

**Theorem C.** *$d_H$ is a metric on $\mathcal{S}(K)$ and hence $(\mathcal{S}(K), d_H)$ is a metric space.*

We call $d_H$ the *Hausdorff metric* to pay homage to Felix Hausdorff.

**Exercise 6.** In this exercise, we study the metric space $(\mathcal{S}(K), d_H)$. First, to get a feel for it, please do the following:

1. Consider the shapes
   $$A = \overline{N_1((0,0))} \qquad \text{and} \qquad B = \overline{N_1((x,0))}$$
   in $\mathbb{R}^2$ where, say, $K = [-10, 10]^2$ and $-5 \leq x \leq 5$. For your own curiosity, you should draw various instances of $A$ and $B$ and observe that they are the closed unit disks centered at $(0,0)$ and $(x,0)$, respectively. Prove that $d_H(A, B) = |x|$.
2. Now that you have some understanding of $d_H$ at least for simple shapes in $\mathbb{R}^2$, our next aim is to return to the general setting to prove that $(\mathcal{S}(K), d_H)$ is a complete metric space. To this end, first prove the following general lemma:

   **Lemma D.** *Let $\{p_n\}$ be a Cauchy sequence in a metric space $(X, d)$. Then, $\{p_n\}$ has a super-duper Cauchy subsequence, i.e., there is a subsequence $\{p_{n_k}\}$ for which*

   $$d(p_{n_k}, p_{n_{k+1}}) \leq 2^{-k}$$

   *for each $k = 1, 2, \ldots,$.*

3. Now, let $\{A_n\}$ be a sequence of shapes in $K \subseteq \mathbb{R}^d$ which is Cauchy in the Hausdorff metric. In view of the lemma you proved above, let $B_k = \{A_{n_k}\}$ be a super-duper Cauchy subsequence in $d_H$. Show that, there exists a sequence $\{x_k\}$ in $K$ for which $x_k \in B_k$ for each $k$ and $|x_k - x_{k+1}| \leq 2^{-k}$ for $k = 1, 2, \dots$.

4. By what you just proved, $\{x_k\}$ is a Cauchy and, since it is a Cauchy sequence in a compact region $K$ in $\mathbb{R}^d$, it converges to some $x$. Show that
$$|x - x_k| \leq 2^{1-k}$$
for all $k$. Hint: First prove that, for any $j \geq 1$,
$$|x_{k+j} - x_k| \leq 2^{-k} + 2^{-k-1} + \cdots + 2^{-k-j} \leq 2^{-k}(1 + 1/2 + 1/2^2 + \cdots 1/2^j) \leq 2^{-k} \times 2 = 2^{1-k}$$
for each $k$. Then take a limit.

5. Let's now denote by $B$ the set of $x \in K$ for which $|x - x_k| \leq 2^{1-k}$ for some sequence $\{x_k\}$ in $K$ having $x_k \in B_k$ for all $x$. By what we showed above, $B$ is non-empty and necessarily bounded by $K$. Consequently, its closure $A = \overline{B}$ is non-empty and compact. Hence $A$ is a shape! Prove that
$$A \subseteq N_{2^{1-k}}(B_k)$$
for each $k$.

6. Prove also that
$$B_k \subseteq N_{2^{1-k}}(A)$$
and conclude that
$$d_H(A, B_k) \leq 2^{1-k}$$
for every $k$.

7. Conclude that $\{B_k\}$ converges to $A$ in the $d_H$ metric and use it to finish the proof that $(\mathcal{S}(K), d_H\}$ is complete.