As promised, below is an itemized list of topics ahead of your second midterm examination. As this course is cumulative by nature, you should not ignore the material from before Midterm 1 (and it might be helpful to look back on it). Still, the exam will focus on the material in the course notes from the section on Power series solutions to the end of Section 4.4 in the book. As the course notes are a little thin in Chapter 4 of the book, below I have also provided you with relevant portions of reading in the excellent text by Boyce and DiPrima that we have discussed often. In studying for Thursday's midterm exam, please note that I consider the homework exercises and the examples I've covered in lecture to be the best source of practice problems. If you know how to approach each problem, are able to work quickly and accurately, and understand the theory and methodology by which you have obtained a solution, you should perform well on the exam.

Theory

1. You should know that the Cauchy-Kovalevskaya theorem is what justifies our method of power series solutions: That is, if we have a linear differential equations with analytic coefficients, we know that solutions will also be analytic (and therefore have power series representations).

2. You should know (and know how to prove Proposition 3.6.1) which guarantees that all solutions to an inhomogeneous linear ode are given by $y_h + y_p$ where y_p is simply one (any) particular solutions.

3. In Section 4.1, you should understand the set up of general first-order $n \times n$ systems, spaces of the type $C^k(I, \mathbb{R}^n)$, and you should understand (well!) the P-L theorem for $n \times n$ first order systems. Note: In checking if a solution $\mathbf{x}(t)$ is unique, the hypotheses ask nothing of $\mathbf{x}(t)$, but of $\mathbf{F}(t, \mathbf{x})$ as a function of t and $\mathbf{x}$ as independent variables. Could you cook up your own 2×2 first order system, solutions to it, and verify that for given initial conditions, a solution is unique? Often, folks struggle with the vector formalism here and that's why we spent a lot of time on it. You should be at the point where you feel comfortable with it and using this formalism.

4. You should understand Exercise 45 and you should know it well (especially in the $n = 2$ case which I explicitly did in lecture). As we discussed, the point of this exercise is to show that there is a one-to-one correspondence between nth-order scalar equations (and their initial value problems) and certain first-order $n \times n$ systems (and their initial value problems). You should know how a solution of one maps uniquely to a solution of the other and vice versa. It's good to have examples in mind if the general formulation is unclear at first. Finally, you should know how the first-order system P-L theorem recaptures all of the previous P-L theorems. This is Parts 3 and 4 of this exercise.

5. You should know the general theory if first-order $n \times n$ linear systems covered by Theorem 4.2.1. Note, we took a full lecture and a half to prove this theorem in class and it's important that you understand it and how it can be used. Note that Exercise 46 is dedicated to this argument as well (and deals with the Wronskian). Also, as you study this, you should pay attention to the similarities between this theory and the linear theory for homogeneous second-order scalar equations (e.g. with $L[y] = y'' + py' + qy$).

6. You should understand Proposition 4.3.1 and its proof.

7. For linear constant-coefficient systems, you should know the content of Exercise 48 (which we covered seriously in lecture). Note: The big take away is that, if you have a matrix A with n distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ with corresponding eigenvectors $v_1, v_2, \ldots, v_n \in \mathbb{R}^n$, then

$$\mathbf{x}_1(t) = e^{\lambda_1 t} v_1,\ \mathbf{x}_2(t) = e^{\lambda_2 t} v_2, \ldots, \mathbf{x}_n(t) = e^{\lambda_n t} v_n$$

form a basis for the solution space to $\dot{\mathbf{x}} = A\mathbf{x}$.

8. You should know Proposition 4.3.3 and how to apply it. We will do this on Monday, November 17th – but you should read it now.

9. In class we covered (briefly) the matrix exponential. That is, given an $n \times n$ matrix A, this is the $n \times n$ matrix-valued function defined by the convergent power series

$$e^{tA} = I + tA + \frac{t^2}{2}A^2 + \frac{t^3}{3!}A^3 + \cdots = \sum_{k=0}^{\infty} \frac{t^k}{k!}A^k$$

As we discussed in class, this matrix-valued function of t has the property that

$$e^{0A} = I \qquad \text{and} \qquad \frac{d}{dt}e^{tA} = Ae^{tA}$$

for every time t. Note: I will not ask you to verify the above however, you should know that the convergence actually does work very well, showing it requires some non-trivial analysis that is beyond the scope of this class. For the matrix exponential, we covered a few nice examples:

(a) If $A = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$, then $e^{tA} = \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{-t} \end{pmatrix}$. We computed this in class and you should check it.

(b) More generally, if $D = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ is an $n \times n$ diagonal matrix, then

$$e^{tD} = \begin{pmatrix} e^{\lambda_1 t} & 0 & \ldots & 0 \\ 0 & e^{\lambda_2 t} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & e^{\lambda_n t} \end{pmatrix}$$

(c) We also showed that, if A was a diagonalizable matrix (with $A = M^{-1}DM$ for some diagonal matrix $D = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and an invertible matrix M), then

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k}{k!}(M^{-1}DM)^k = M^{-1}\left(\sum_{k=0}^{\infty} \frac{t^k}{k!}D^k\right)M = M^{-1}e^{tD}M = M^{-1}\begin{pmatrix} e^{\lambda_1 t} & 0 & \ldots & 0 \\ 0 & e^{\lambda_2 t} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & e^{\lambda_n t} \end{pmatrix}M =$$

(d) In fact, on Friday, we computed the above for

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{and where } M = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix},$$

$$e^{tA} = \begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix}$$

where

$$\cosh(t) = \frac{e^t + e^{-t}}{2} \qquad \text{and} \qquad \sinh(t) = \frac{e^t - e^{-t}}{2}.$$

With this, we easily verified that

$$e^{0A} = \begin{pmatrix} \cosh(0) & \sinh(0) \\ \sinh(0) & \cosh(0) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\frac{d}{dt}e^{tA}\Big|_{t=0} = \begin{pmatrix} \sinh(t) & \cosh(t) \\ \cosh(t) & \sinh(t) \end{pmatrix}\Big|_{t=0} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = A,$$

as required.

(e) Finally, we observed that, given a matrix A, we can produce a basis for the solution space to $\dot{\mathbf{x}} = Ax$ by simply setting

$$\mathbf{x}_1(t) = e^{tA}\mathbf{e}_1,\ \mathbf{x}_2(t) = e^{tA}\mathbf{e}_2, \dots, \mathbf{x}_n(t) = e^{tA}\mathbf{e}_n$$

where $\mathbf{e}_1, \mathbf{e}_2 \dots \mathbf{e}_n$ are the standard basis vectors of $\mathbb{R}^n$ but, in fact, could be any basis of $\mathbb{R}^n$.

Overall, I would like you to be familiar with the ideas behind what we did for this (and be comfortable with some of the calculations). Because we did not treat a general theory, I'm not asking you to know any results precisely only that these matrix exponentials can be computed and they lead to solutions to linear constant-coefficient systems. More on this can be found in Section 7.8 in Boyce and DiPrima's book.

Procedures/Solution Methods

1. You should be able to solve second-order linear equations via power series. In particular, if $p(t)$ and $q(t)$ are analytic functions (say, analytic at 0, e.g., polynomials), then you should know how to use power series methods to produce a solution of the form

$$y(t) = \sum_{n=0}^{\infty} c_n t^n$$

to the differential equation

$$y'' + p(t)y' + q(t)y = 0.$$

In doing this, you should understand that linearly independent solutions can be produced by specifying the zeroth and first coefficients c_0 and c_1. You should understand fully Examples 9 and 10 and Exercises 37 and 38. More examples can be found in Sections 5.2 and 5.3 of Boyce and DiPrima's text.

2. Given a sufficiently nice function $r(t)$, you should know the theory for solutions to the inhomogeneous differential equation

$$y'' + p(t)y' + q(t)y = r(t).$$

You should know that, if you can solve the corresponding homogeneous equation ($y'' + py' + qy = 0$ and find a basis of solutions y_1 and y_2 to it), then knowing ANY solution y_p (called "y particular) to the inhomogenous problem is enough to produce all solutions. In fact, you should know (and know why) all solutions to the inhomogenous problem are given by

$$y(t) = C_1 y_1(t) + C_2 y_2(t) + y_p(t)$$

for constants C_1, C_2. Relevant here are Proposition 3.6.1. and Corollary 3.6.2.

3. You should understand that power series solution methods can also help produce particular solutions to inhomogeneous problems. Relevant here is Exercise 40.

4. You should know (and be very comfortable) applying the method of undetermined coefficients. Further, you should be comfortable using the "trick" by which, in the case that a "guess" is already a member of the kernel, you multiply the full guess by t. Relevant here is Example 15 and Exercise 41 (and the principle in between).

5. You should understand the applications in Section 3.8. Note: I will not be asking you physics problems directly, but we did cover a sufficient amount of material in this that you should have a good feel for what's going on physically. More importantly, you should understand the mathematics completely.

6. Given an $n \times n$ first order system $\dot{\mathbf{x}} = F(t, \mathbf{x})$ and a function $\mathbf{x} \in C^1(I; \mathbb{R}^n)$, you should be able to verify that $\mathbf{x}(t)$ is a solution to your equation.

7. For a linear first-order $n \times n$ system of the form $\dot{\mathbf{x}} = P(t)\mathbf{x}$, and n solutions $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$, could you verify that they form a basis for the solution space to the equation? This might involve considering the Wronskian.

8. Given a 2×2 matrix A, you should be able to find A's eigen vectors and eigen values. If it has distinct eigenvalues λ_1 and λ_2 with corresponding eigenvectors v_1 and v_2, you should be able to use these to produce a basis of solutions to the first-order 2×2 system $\dot{\mathbf{x}} = A\mathbf{x}$. See Example 1 on Page 134. More examples/exercises can be found in Section 7.5 of Boyce and DiPrima.

9. In applying Proposition 4.3.3, you should understand Example 2 on Page 136. More examples can also be found in Section 7.6 (and 7.7) of Boyce and DiPrima. We will also work through this example on Monday.