As promised, I have written an itemized list of topics we've covered in Math 311 since the beginning of the semester. This list of topics (and the proportion of time we've spend on them since the beginning of the semester) will align with the problems you will see on Wednesday's midterm exam. In studying for Wednesday's midterm exam, please note that I consider the homework exercises and the examples I've covered in lecture to be the best source of practice problems. If you know how to approach each problem, are able to work quickly and accurately, and understand the theory and methodology by which you have obtained a solution, you should perform well on the exam. If, after exhausting all of these resources, you're still looking for extra problems to solve, take a look at any edition of the Boyce-Di Prima book referenced in the course notes.

Theory

1. You should know all definitions (precisely). In particular, you should know the definitions of the following terms:

   a. ordinary differential equation (and the definitions of the following classes of first-order ODEs)
      - separable differential equation
      - autonomous differential equation
      - homogeneous linear first-order differential equation
      - inhomogeneous linear first-order differential equation
      - exact equation

   b. order of ode

   c. initial condition

   d. initial point

   e. initial value problem

   f. continuously differentiable function

   g. analytic solution

   h. integral curve

   i. equilibrium solution

   j. equilibrium value

   k. sink/source/node

   l. Euler's algorithm/method for numerical approximation of solutions to first order IVPs.

   m. Second-order linear differential equation

   n. Second-order linear homogeneous differential equation

   o. Second-order linear homogeneous constant-coefficient differential equation

   p. Fundamental generating set of solutions

   q. For an interval I, the spaces $C^0(I)$, $C^1(I)$ and $C^2(I)$.

   r. For an interval I and natural number n, the space of polynomials $\mathcal{P}_n(I)$.

   s. Vector space, subspace

   t. Linearly independent set, spanning set, basis

   u. Linear operator, isomorphism

   v. Kernel of linear operator

   w. Wronskian matrix

   x. Wronskian determinant

2. Please know all of the major theorems and propositions we have studied so far in this course; this, in particular, means having a solid understanding of each theorem's/proposition's hypotheses. If I have discussed (or you have explored in homework) the necessity of a certain hypothesis, you should understand why. The list of major results is as follows:

   (a)

   Proposition 1. *For the differential equation*

   $$\frac{dy}{dt} = f(t, y),$$

   the constant function $y(t) = y_0$ *is an equilibrium solution if and only if the number* y_0 *is an equilibrium value.*

   (b)

   Proposition 2. *Let* y_0 *be an equilibrium value for the autonomous differential equation*

   $$\frac{dy}{dt} = h(y).$$

   and suppose that $h(y)$ *is differentiable at* y_0.

   i. If $\frac{\partial h}{\partial y}(y_0) < 0$*, then* y_0 *is a sink.*

   ii. If $\frac{\partial h}{\partial y}(y_0) > 0$ *then* y_0 *is a source.*

   iii. If $\frac{\partial h}{\partial y}(y_0) = 0$*, nothing can be said and further analysis is required.*

   (c)

   Theorem 3 (Picard-Lindelöf)**.** *Let* $f = f(t, y)$ *be defined, continuous and have continuous partial derivative* $\partial f/\partial y$ *on the rectangle*

   $$R = \{(t, y) : \alpha \leq t \leq \beta, \gamma \leq y \leq \kappa\} = [\alpha, \beta] \times [\gamma, \kappa];$$

   here $\alpha, \beta, \gamma, \kappa$ *are real numbers such that* $\alpha < \beta$ *and* $\gamma < \kappa$*. If* (t_0, y_0) *sits in the interior of the rectangle* R*, i.e.,* $\alpha < t_0 < \beta$ *and* $\gamma < y_0 < \kappa$*, then the differential equation*

   $$\frac{dy}{dt} = f(t, y)$$

   has a unique solution $y(t)$ *passing through the initial point* (t_0, y_0)*. More precisely, there is one and only one function* $y(t)$ *satisfying the initial value problem*

   $$\begin{cases} \frac{d}{dt}y(t) = f(t, y(t)) \qquad y(t_0) = y_0 \end{cases}$$

   for all $t \in (t_0 - \delta, t_0 + \delta)$*; here,* δ *is some positive number and* $y(t)$ *is necessarily continuous and differentiable and has a continuous derivative on the interval* $(t_0 - \delta, t_0 + \delta)$*.*

   (d)

   Theorem 4. *Let* $a(t)$ *and* $b(t)$ *be continuous functions on an interval* $I = (\alpha, \beta)$ *and let* $A(t)$ *be an antiderivative of* $a(t)$*. Then the differential equation*

   $$\frac{dy}{dt} + a(t)y = b(t)$$

   has infinitely many solutions, all of which are given by

   $$y(t) = \frac{1}{\mu(t)} \int \mu(t)b(t)\,dt + \frac{C}{\mu(t)}$$

   for $t \in I$ *where* C *is a constant and* $\mu(t) = e^{A(t)}$*.*

3. You should know how to apply the Picard-Lindelöf Theorem to the context of solutions curves through initial points. You should know where the conclusions of this theorem break down when the hypotheses aren't met. Specifically, you should know all of the ideas surrounding Examples 15, 16, and 17 and Exercises 14, 15, and 16.

4.

Theorem 5. *Let the function M, N, $\partial M/\partial y$ and $\partial N/\partial x$ be continuous on the rectangle*

$$R = \{(x, y) \in \mathbb{R}^2 : a < x < b, c < y < d\}$$

where a, b, c and d are such that $a < b$ and $c < d$ (we allow the possibility that $a = c = -\infty$ and $b = d = \infty$). Then

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0$$

is an exact equation (in R) if and only if

$$\frac{\partial M}{\partial y}(x, y) = \frac{\partial N}{\partial x}(x, y)$$

at each point in R.

5.

Theorem 6. *Let I be an open interval and let p, q and r be continuous functions on I. Then, given any $t_0 \in I$ and any y_0 and $y_0' \in \mathbb{R}$, the initial value problem*

$$\begin{cases} y'' + p(t)y' + q(t)y = r(t) \\ \\ y(t_0) = y_0 \\ y'(t_0) = y_0' \end{cases}$$

has a unique solution y. Furthermore, this solution is twice continuously differentiable on the entire interval I.

6.

Proposition 7 (The Principle of Superposition)**.** *Let y_1 and y_2 be two solutions to*

$$y'' + py' + qy = 0 \tag{1}$$

where p and q are continuous functions on an interval I. Then, for any constants C_1 and C_2,

$$y(t) = C_1 y_1(t) + C_2 y_2(t)$$

is a solution to (1).

7.

Theorem 8. *Let y_1 and y_2 be solutions to* (1)*. Given $t_0 \in I$, if*

$$w_{y_1,y_2}(t_0) = y_1(t_0)y_2'(t_0) - y_2(t_0)y_1'(t_0) \neq 0,$$

or equivalently, if the Wronskian matrix at t_0 is invertible, then, for any real numbers y_0 and y_0', the initial value problem

$$\begin{cases} y'' + p(t)y' + q(t)y = 0 \\ \\ y(t_0) = y_0 \\ y'(t_0) = y_0' \end{cases}$$

is solved by

$$y(t) = C_1 y_1(t) + C_2 y_2(t)$$

where

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix}^{-1} \begin{pmatrix} y_0 \\ y_0' \end{pmatrix} = (W_{y_1,y_2}(t_0))^{-1} \begin{pmatrix} y_0 \\ y_0' \end{pmatrix}.$$

8.

Theorem 9. *Suppose that y_1 and y_2 solve the differential equation* (1)*. If*

$$w_{y_1,y_2}(t) = y_1(t)y_2'(t) - y_2(t)y_1'(t) \neq 0 \quad \text{for some } t \in I \tag{2}$$

(any t will do!), then

$$y(t) = C_1 y_1(t) + C_2 y_2(t) \tag{3}$$

is a general solution to (1)*. More precisely, if the condition* (2) *is satisfied (that is for any t whatsoever in I), then the Wronskian matrix $W_{y_1,y_2}(t)$ is invertible for all $t \in I$ and so any initial value problem of the form*

$$\begin{cases} y'' + p(t)y' + q(t)y = 0 \\ \\ y(t_0) = y_0 \\ y'(t_0) = y_0' \end{cases} \tag{4}$$

where $t_0 \in I$ and $y_0, y_0' \in \mathbb{R}$ can be solved by (3) *by simply putting*

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = (W_{y_1,y_2}(t))^{-1} \begin{pmatrix} y_0 \\ y_0' \end{pmatrix} = \begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix}^{-1} \begin{pmatrix} y_0 \\ y_0' \end{pmatrix}.$$

9. The following result guarantees existence of a fundamental generating set.

Theorem 10. *Consider the linear homogeneous second-order differential equation* (1)*. There exists two solutions y_1 and y_2 for which the conclusion of Theorem 9 holds. In other words, there are two functions y_1 and y_2 such that*

$$y(t) = C_1 y_1(t) + C_2 y_2(t)$$

is a general solution to the above differential equation.

10. The following proposition tells you how to obtain a second "good" solution given that you already know one.

Proposition 11. *Let y_1 be a non-zero solution* (1) *let y solve the first-order linear differential equation*

$$y_1(t)y' - y_1'(t)y = e^{-P(t)}$$

for $t \in I$ where $P' = p$. Then $y_2 = y$ also solves (1) *(and so is twice continuously differentiable on I) and furthermore the pair $\{y_1, y_2\}$ is a fundamental generating set of solutions for* (1)*.*

11.

Proposition 12. *The map $L : C^2(I) \to C^0(I)$ is a linear operator from the vector space $C^2(I)$ of twice continuously differentiable functions into the vector space of continuous functions on the interval I. The kernel of this operator* $\ker(L)$ *is precisely the set of solutions to the homogeneous differential equation* (1)*.*

12.

Proposition 13. *Let $y_1, y_2 \in \ker(L)$. Then y_1 and y_2 are linearly independent if and only if*

$$w_{y_1,y_2}(t) \neq 0$$

for some (and hence all) $t \in I$.

13.

Corollary 14. *Given any two elements $y_1, y_2 \in \ker(L)$, the following are equivalent.*

(a) y_1 *and* y_2 *form a fundamental generating set of solutions to* (1).

(b) y_1 *and* y_2 *form a basis for* $\ker(L)$.

(c) y_1 *and* y_2 *are linearly independent.*

(d) $w_{y_1,y_2}(t) \neq 0$ *for some* $t \in I$.

(e) $w_{y_1,y_2}(t) \neq 0$ *for all* $t \in I$.

14. Now we switch our focus to constant-coefficient linear homogeneous differential equations. The following theorem amasses the results we will learn in lecture on Monday.

Theorem 15. *Let $b, c \in \mathbb{R}$ and consider the constant-coefficient linear differential equation*

$$L[y] = y'' + by' + cy = 0 \tag{5}$$

and its corresponding polynomial equation

$$r^2 + br + c = 0. \tag{6}$$

(a) If $b^2 > 4c$, set

$$r_1 = -\frac{b}{2} + \frac{\sqrt{b^2 - 4c}}{2} \quad \text{and} \quad r_2 = -\frac{b}{2} - \frac{\sqrt{b^2 - 4c}}{2};$$

these are distinct real solutions to (6). *Then $y_1(t) = e^{r_1 t}$ and $y_2(t) = e^{r_2 t}$ form a fundamental generating set of solutions to* (5).

(b) If $b^2 = 4c$, set $r = -b/2$ (the unique solution to (6)*, a root of multiplicity* 2*). Then $y_1(t) = e^{rt}$ and $y_2(t) = te^{rt}$ form a fundamental generating set of solutions to* (5).

(c) If $b^2 < 4c$, set

$$\alpha = -\frac{b}{2} \quad \text{and} \quad \beta = \frac{\sqrt{4c - b^2}}{2}$$

and $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$ are distinct complex solutions to (6). *In this case, $e^{r_1 t}$ and $e^{r_2 t}$ form a fundamental generating set of solutions to* (5). *Equivalently $y_1(t) = e^{\alpha t}\cos(\beta t)$ and $y_2(t) = e^{\alpha t}\sin(\beta t)$ form a fundamental generating set of solutions to* (5).

Procedures/Solution Methods

1. You should know the solution methods (general solutions and solutions to initial value problems) for the following classes of equations:

   a. separable equations

   b. autonomous equations

   c. linear equations

2. You should know Newton's Law of Cooling (and how the differential equation is solved).

3. Beyond Newton's law of cooling, you should know the other applications we've covered. This includes radioactive decay and the examples in Chapter 1.

4. You should know how to draw slope fields and qualitatively analyze differential equations using slope fields.

5. Continuing on the topic of the previous item, given an autonomous differential equation
$$\frac{dy}{dt} = f(y),$$
you should be able to do fixed point analysis and qualitative analysis of solutions using the graph of f and the corresponding slope fields.

6. You should know all of the theory above and understand any proofs I have given in class.

7. You should know all the homework exercises.

8. You should know how to check if first-order equations are exact.

9. You should know how to solve exact equations.

10. You should know how to run Euler's method to numerically approximate the solutions to first-order IVPs. In other words, if I gave you a first-order IVP, you should be able to produce (or write down something a calculator could compute) the t_ks and y_ks approximating the solution.

11. Given two solutions to a second-order linear homogeneous ODE, you should know how to make use of the theorems above to produce general solutions to solve initial value problems.

12. Given one solution to a linear second-order differential equation, you should know how to make use Abel's identity to find another solution which, together with the first solution, forms a fundamental generating set of solutions.

13. You should know how to check linear independence of functions and of solutions using the Wronskian.

14. You should know all of the homework exercises and be comfortable with the methods of linear algebra behind them. Though the exam won't focus on linear algebra in the abstract, you should be comfortable with the linear algebraic arguments made in class and those used to obtain solutions to the homework.

15. You should know know how to solve any/every second-order constant-coefficient homogeneous linear differential equation and any corresponding initial value problem.

Things not to worry about for this exam:

1. Anything concerning partial differential equations from Chapter 1.

2. Things in the subsection entitled "the error in Euler". I'd like you to have an idea what's going on here, e.g., that it gives sufficient conditions under which Euler's method converges to a solution, but you won't be asked anything else about this on the exam.